Algorithm II

7. Network Flow I

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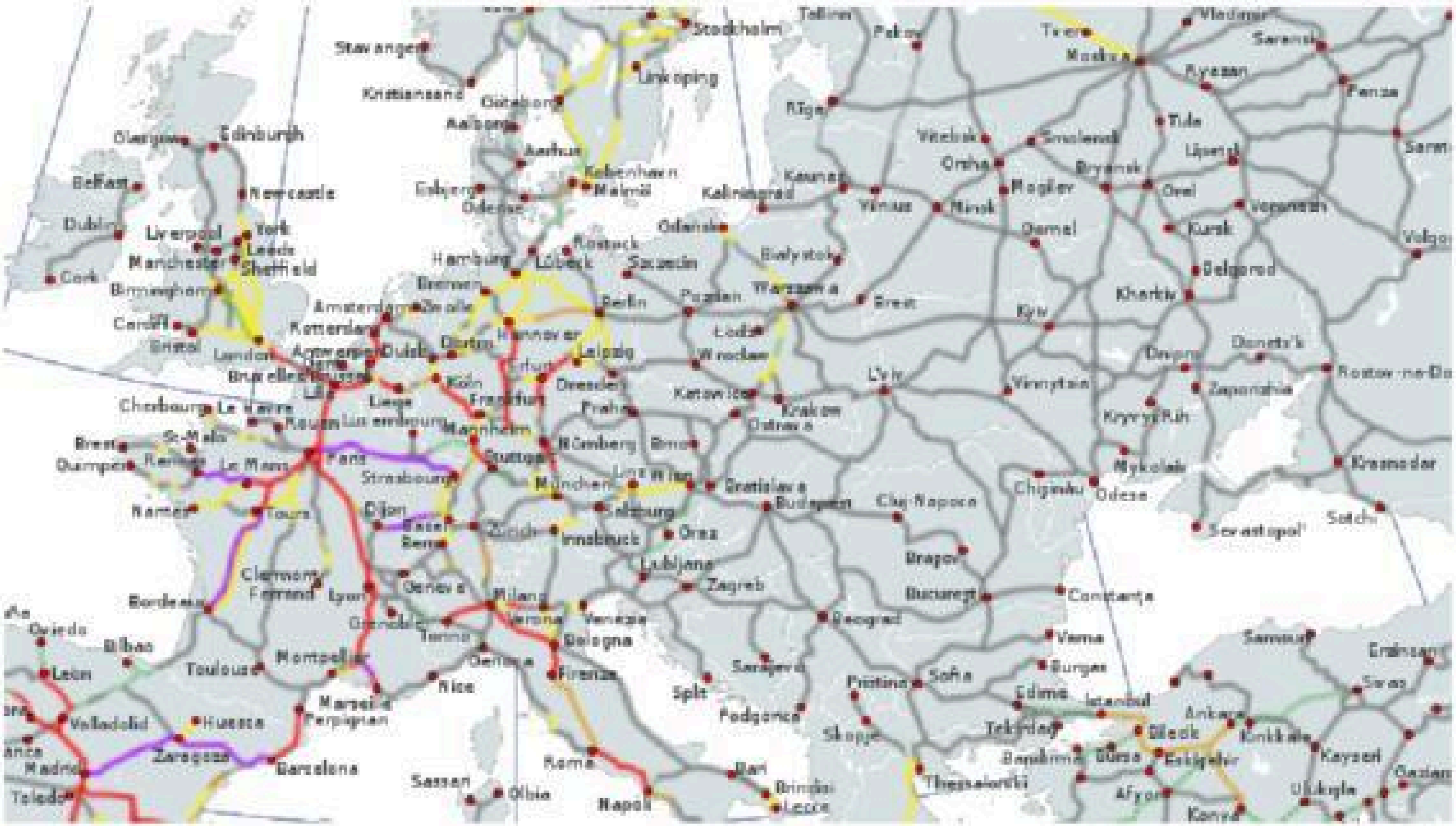
Flow and Cut



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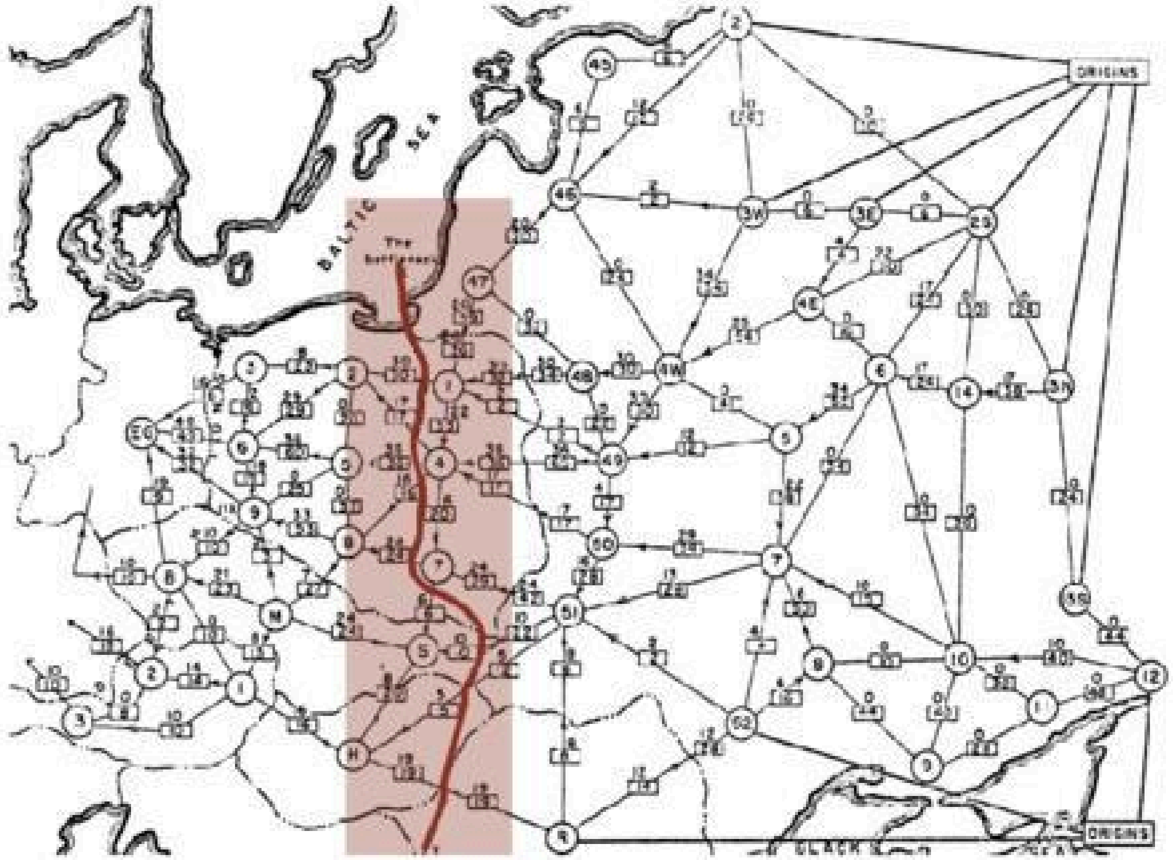
Flow and Cut - supply



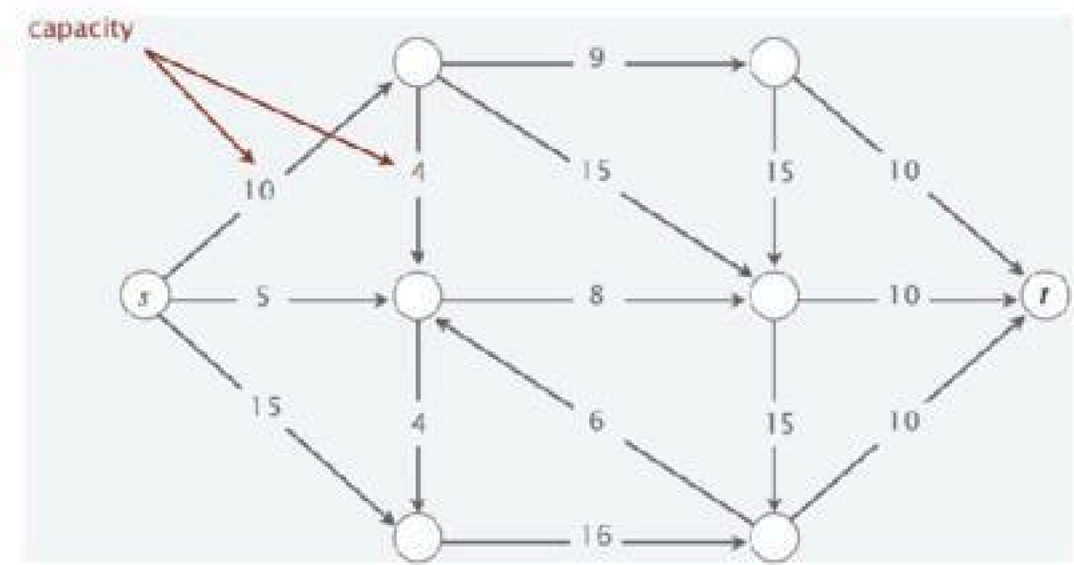
Max-flow and min-cut problems

Flow network

A **flow network** is a tuple $G = (V, E, s, t, c)$.

- Digraph (V, E) with source $s \in V$ and sink $t \in V$.
- Capacity $c(e) \geq 0$ for each $e \in E$.

Intuition. Material flowing through a transportation network; material originates at source and is sent to sink.

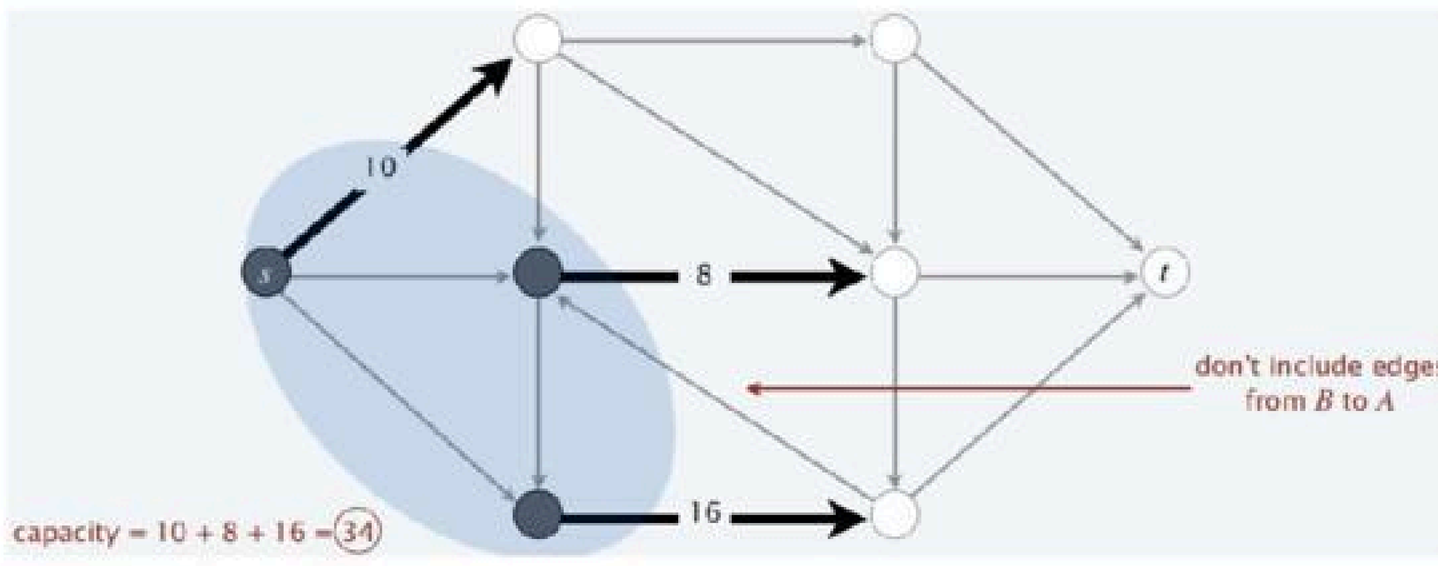


Minimum-cut problem

Def. An **st-cut (cut)** is a partition (A, B) of the nodes with $s \in A$ and $t \in B$.

Def. Its capacity is the sum of the capacities of the edges from A to B.

- $cap(A, B) = \sum_{e \text{ out } A} c(e)$

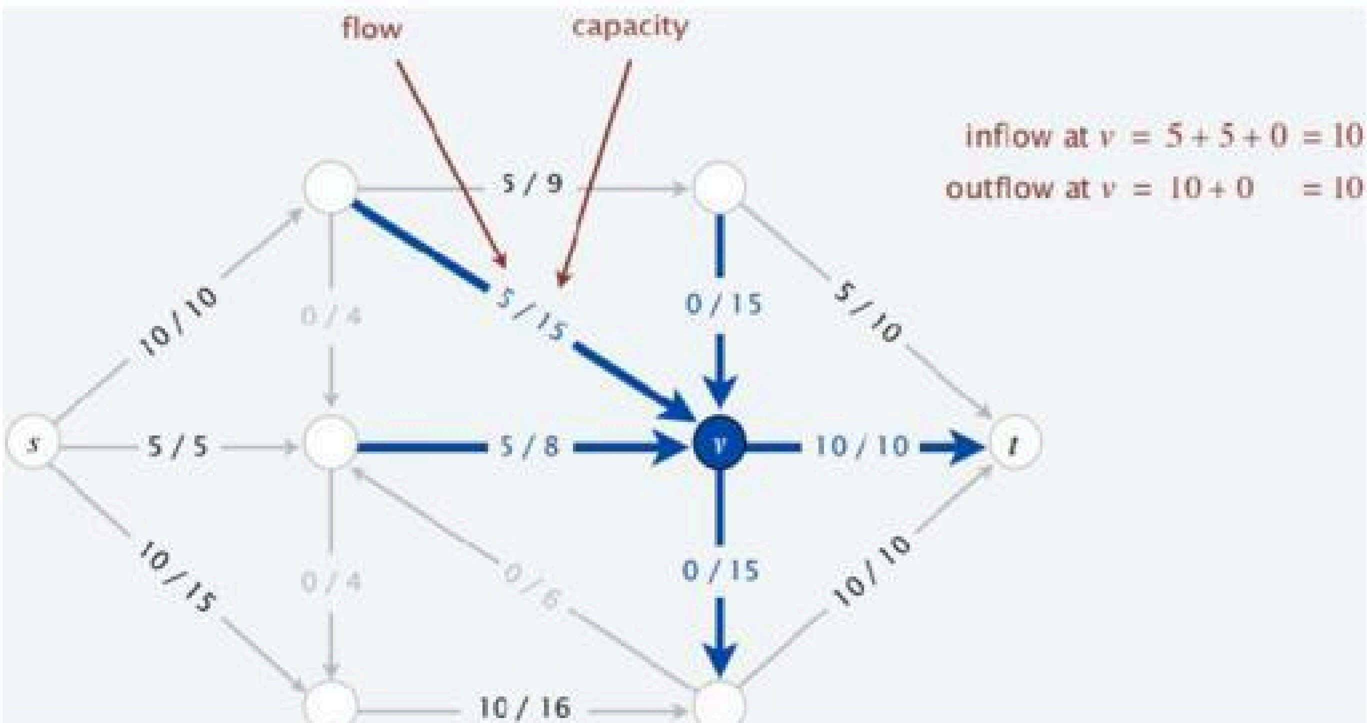


Min-cut problem. Find a cut of minimum capacity.

Maximum-flow problem

Def. An **st-flow (flow)** f is a function that satisfies:

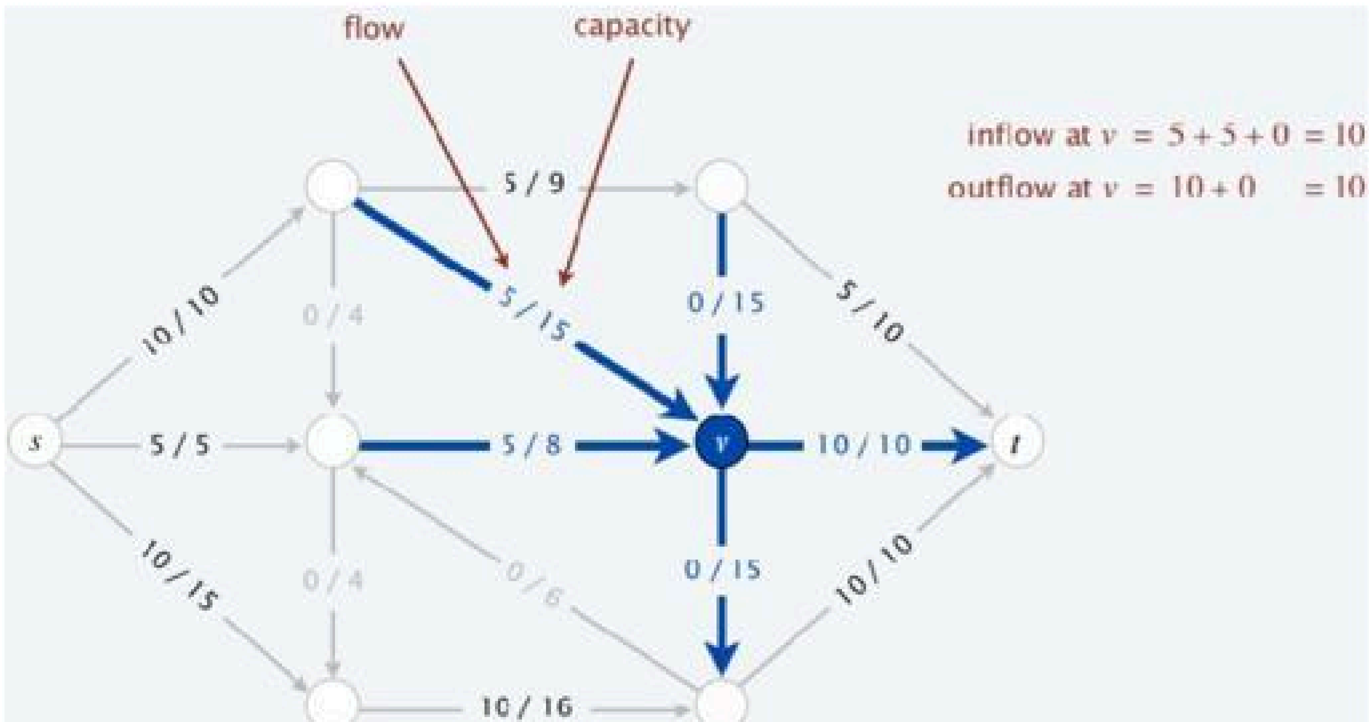
- [**capacity**] For each $e \in E$: $0 \leq f(e) \leq c(e)$
- [**flow conservation**] For each $v \in V - \{s, t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out } v} f(e)$



Maximum-flow problem (cont.)

Def. The **value** of a flow f is: $val(f) = \sum_{e \text{ out } s} f(e) - \sum_{e \text{ into } s} f(e)$

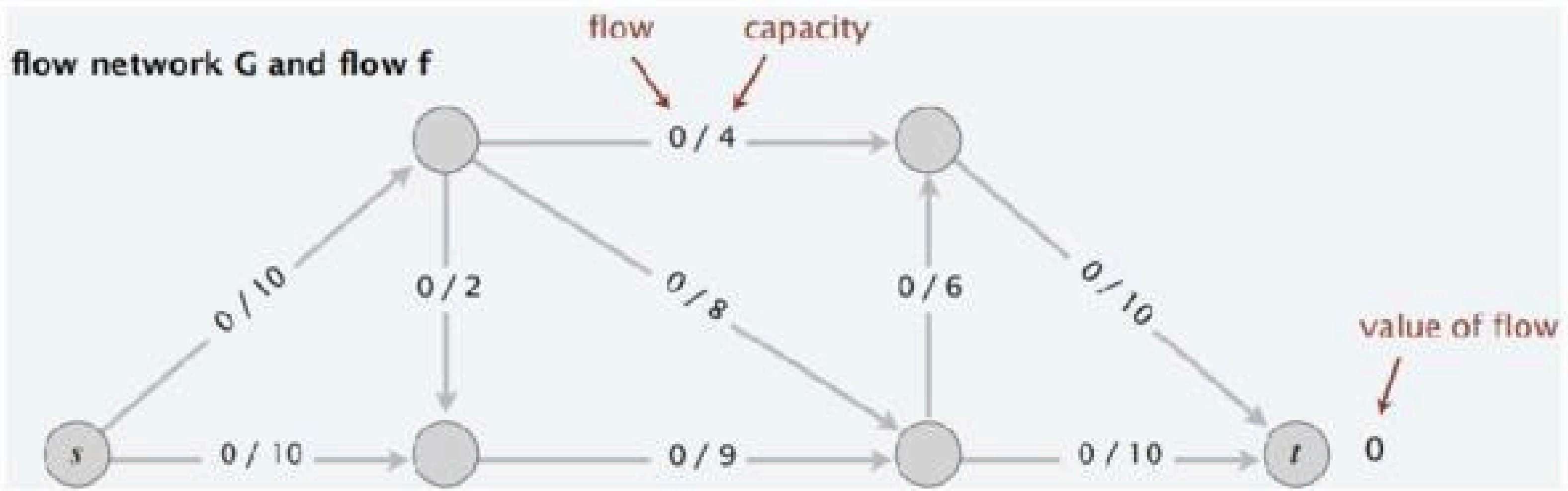
Max-flow problem. Find a flow of maximum value.



Ford-Fulkerson algorithm

Toward a max-flow algorithm

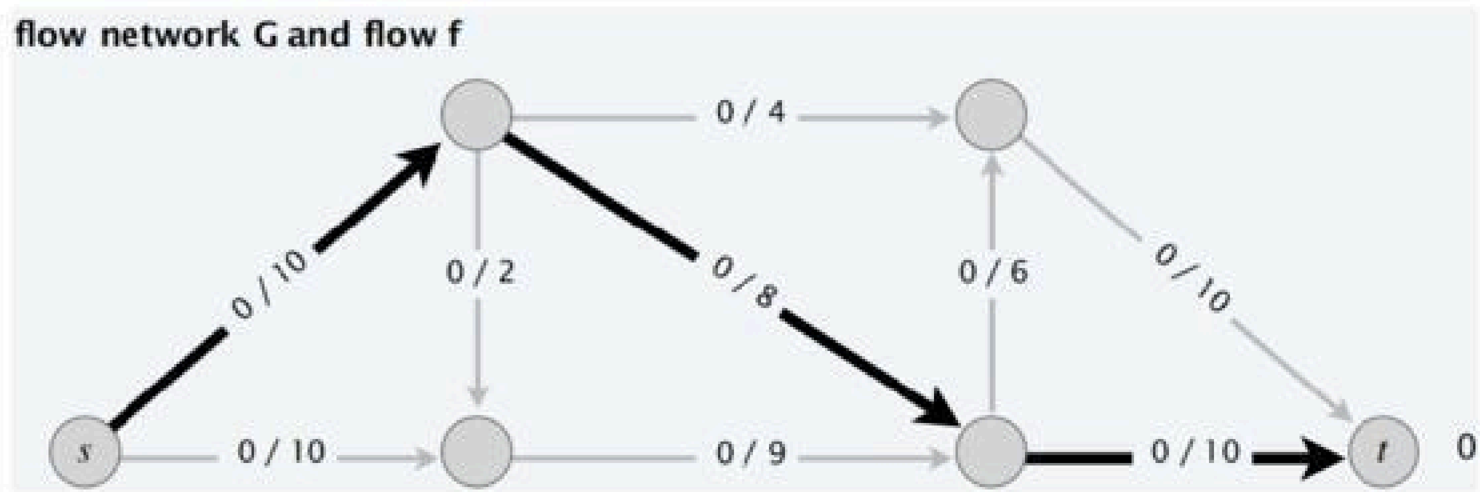
Greedy algorithm.



- Start with $f(e) = 0$ for each edge $e \in E$.

Toward a max-flow algorithm

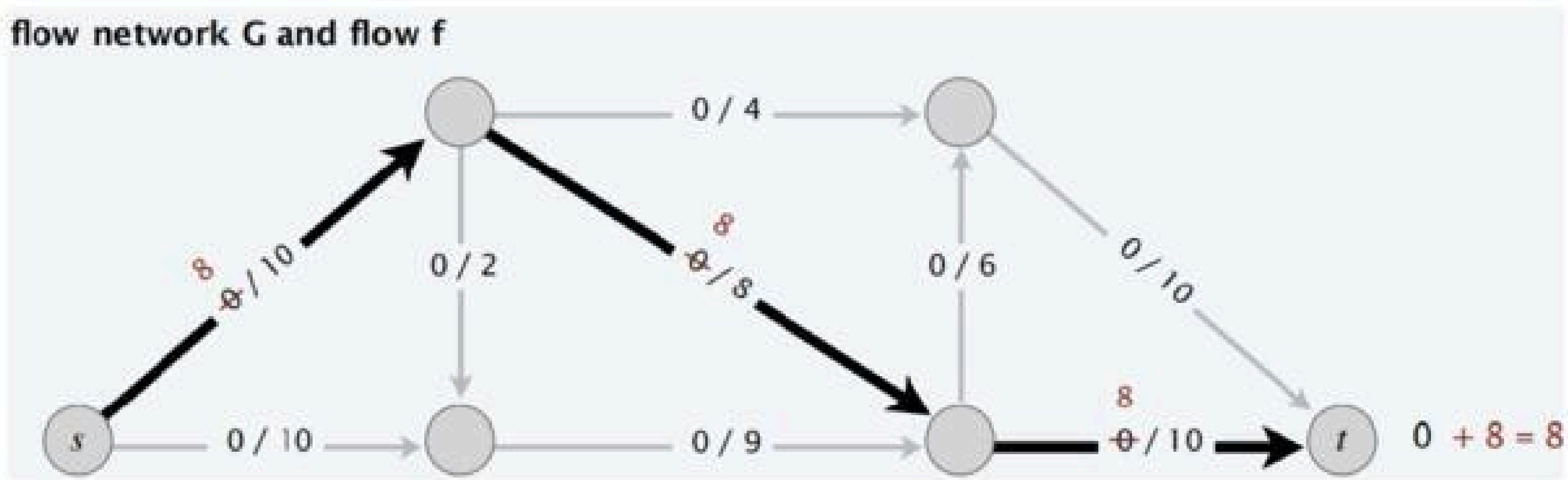
Greedy algorithm.



- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.

Toward a max-flow algorithm

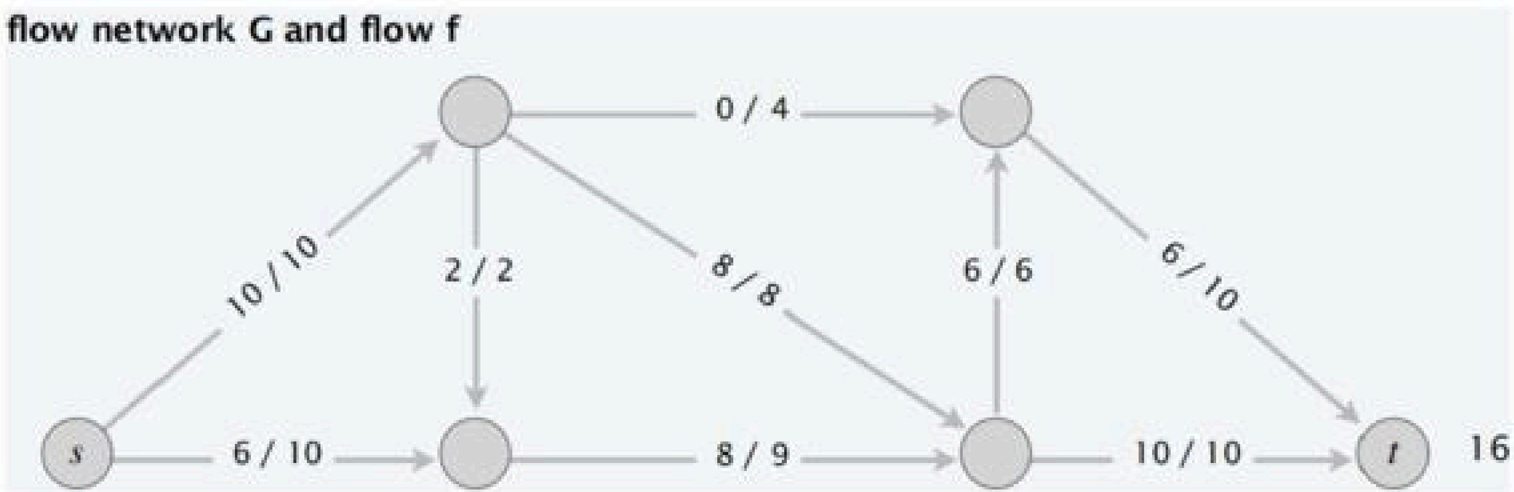
Greedy algorithm.



- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
- Augment flow along path P.

Toward a max-flow algorithm

Greedy algorithm.

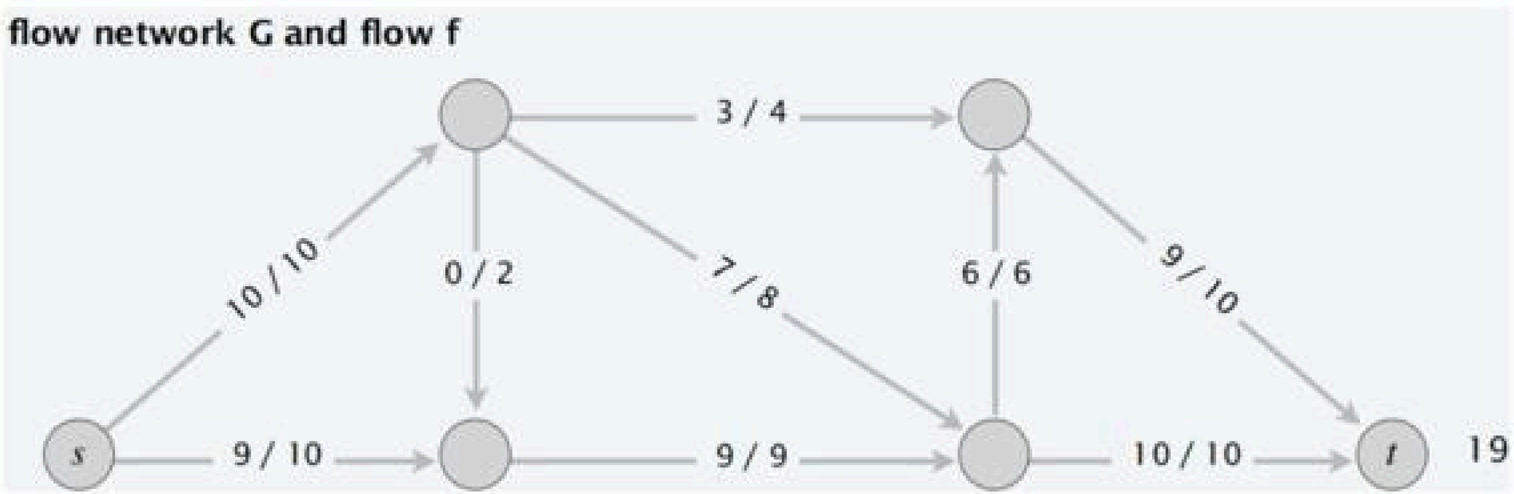


- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
- Augment flow along path P.
- Repeat until you get stuck.

Ending flow value = 16

Toward a max-flow algorithm

Greedy algorithm.



- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
- Augment flow along path P.
- Repeat until you get stuck.

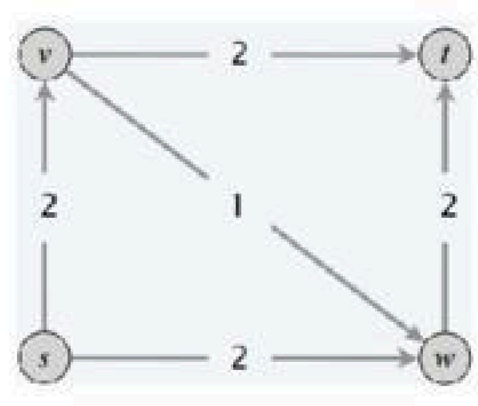
But max-flow value = 19

Why greedy algorithm fails

Q. Why does the greedy algorithm fail?
A. Once greedy algorithm increases flow on an edge, it never decreases it.

Ex. Consider flow network G.

- Max flow f^* has $f^*(v, w) = 0$.
- Greedy algorithm could choose $s \to v \to w \to t$ as first path.



Bottom line. Need some mechanism to “undo” a bad decision.

Residual network

Original edge. $e = (u, v) \in E$.

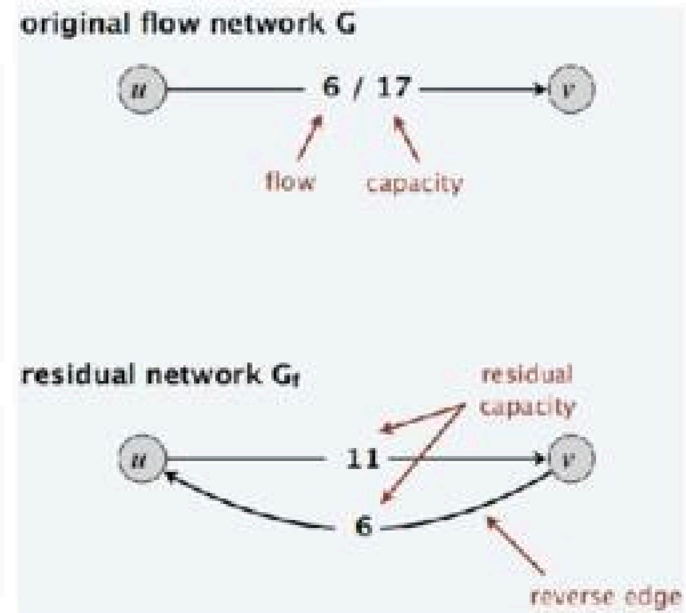
- Flow $f(e)$; Capacity $c(e)$.

Reverse edge. $e^{-1} = (v, u)$.

- "Undo" flow sent.

Residual capacity.

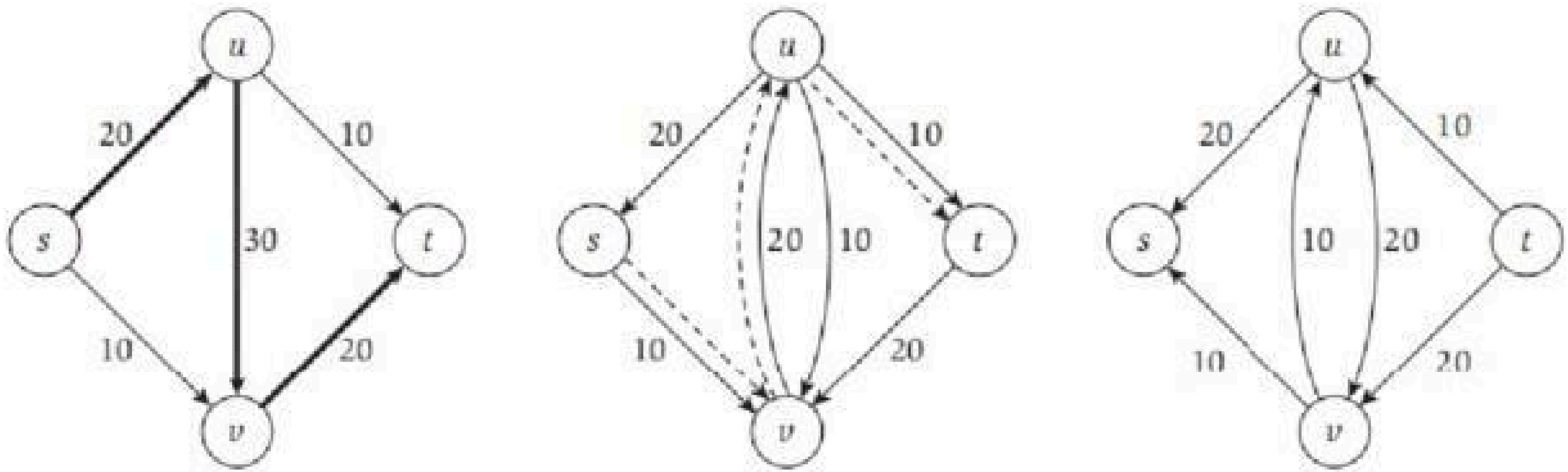
$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if} \quad e \in E \\ f(e) & \text{if} \quad e^{-1} \notin E \end{cases}$$



Residual network (cont.)

Residual network. $G_f = (V, E_f, s, t, c_f)$.

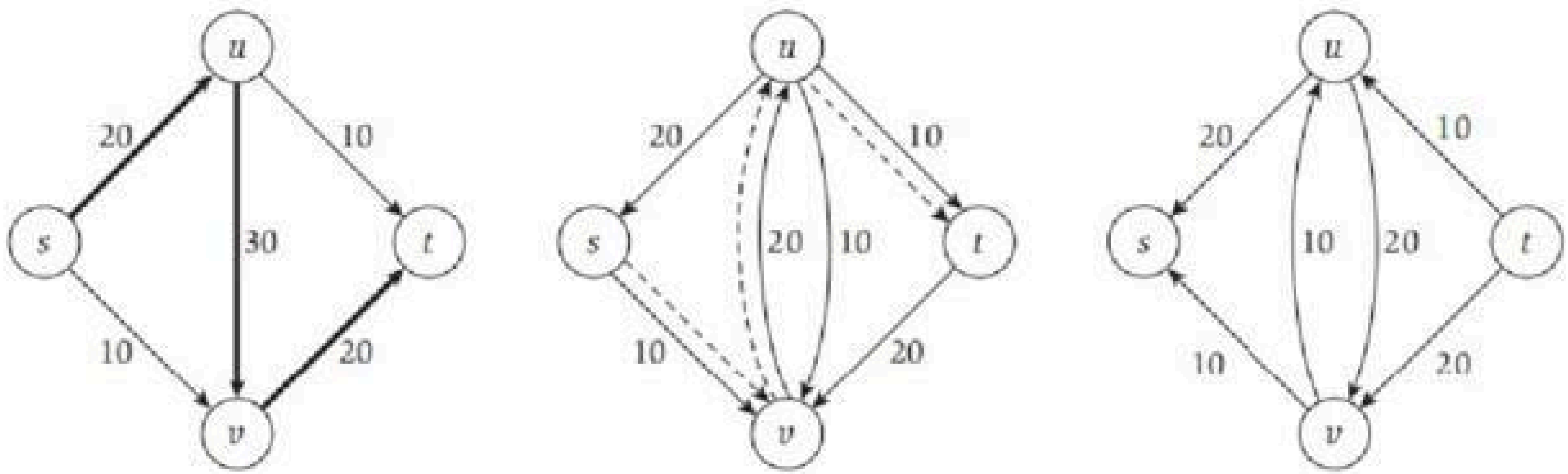
- $E_f = \{e : f(e) < c(e)\} \cup \{e^{-1} : f(e) > 0\}$.
- **Key property**: f' is a flow in G_f iff $f + f'$ is a flow in G.



Augmenting path

Def. An **augmenting path** is a simple $s \rightsquigarrow t$ path in the residual network G_f.

Def. The **bottleneck capacity** of an augmenting path P is the *minimum residual capacity* of any edge in P.



Key property. Let f be a flow and let P be an augmenting path in G_f. Then, after calling f' = AUGMENT(f, c, P), the resulting f' is a flow and $val(f') = val(f) + bottleneck(G_f, P)$.

Augmenting path: algorithm

Key property. Let f be a flow and let P be an augmenting path in G_f. Then, after calling f' = $\texttt{AUGMENT}(f, c, P)$, the resulting f' is a flow and $val(f') = val(f) + bottleneck(G_f, P)$.

$\texttt{AUGMENT}(f, c, P)$

1. Δ = bottleneck capacity of augmenting path P.
2. FOREACH edge $e \in P$:
   1. IF $(e \in E)$ $f(e) = f(e) + \Delta$;
   2. ELSE $f(e^{-1}) = f(e^{-1}) - \Delta$;
3. RETURN f;

Ford-Fulkerson algorithm

Ford-Fulkerson augmenting path algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \leadsto t$ path P in the residual network G_f .
- Augment flow along path P.
- Repeat until you get stuck.

FORD-FULKERSON(G)

1. FOREACH edge $e \in E$: $f(e) = 0$;
2. G_f = residual network of G with respect to flow f;
3. WHILE (there exists an $s \leadsto t$ path P in G_f)
   1. f = AUGMENT(f, c, P);
   2. Update G_f;
4. RETURN f;

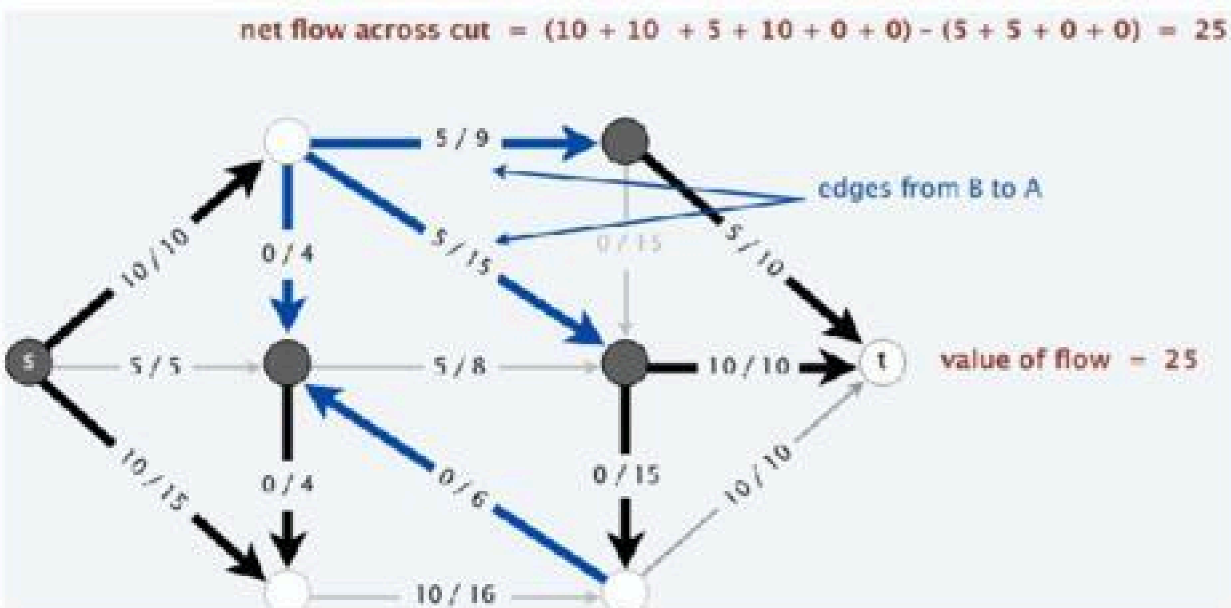
Demo: Ford-Fulkerson

Max-Flow Min-Cut Theorem

Flows and cuts: relationship

Flow value lemma. Let f be any flow and let (A, B) be any cut. Then, the value of the flow f equals the net flow across the cut (A, B).

$$val(f) = \sum_{e \text{ out } A} f(e) - \sum_{e \text{ into } A} f(e)$$



Flows and cuts: relationship (cont.)

Flow value lemma. Let f be any flow and let (A, B) be any cut. Then, the value of the flow f equals the net flow across the cut (A, B).

$$val(f) = \sum_{e \text{ out } A} f(e) - \sum_{e \text{ into } A} f(e)$$

Pf.

$$\begin{aligned} val(f) &= \sum_{e \text{ out } s} f(e) - \sum_{e \text{ into } s} f(e) \\ &= \sum_{v \in A} \Big(\sum_{e \text{ out } v} f(e) - \sum_{e \text{ into } v} f(e) \Big) \\ &= \sum_{e \text{ out } A} f(e) - \sum_{e \text{ into } A} f(e) \end{aligned}$$

Flows and cuts: duality

Weak duality. Let f be any flow and (A, B) be any cut. Then, $val(f) \leq cap(A, B)$.

Pf.

$$\begin{aligned} val(f) &= \sum_{e \text{ out } A} f(e) - \sum_{e \text{ into } A} f(e) \\ &\leq \sum_{e \text{ out } A} f(e) \\ &\leq \sum_{e \text{ out } A} c(e) \\ &= cap(A, B) \end{aligned}$$

Certificate of optimality

Corollary. Let f be a flow and let (A, B) be any cut. If $val(f) = cap(A, B)$, then f is a max flow and (A, B) is a min cut.

Pf.

- For any flow f': $val(f') \leq cap(A, B) = val(f)$.
- For any cut (A', B'): $cap(A', B') \geq val(f) = cap(A, B)$.

Max-flow min-cut theorem I

Max-flow min-cut theorem. [strong duality] Value of a max flow = capacity of a min cut.

Augmenting path theorem. A flow f is a max flow iff no augmenting paths.

Pf. The following three conditions are equivalent for any flow f:

- A. There exists a cut (A, B) such that $cap(A, B) = val(f)$.
- B. f is a max flow.
- C. There is no augmenting path with respect to f.
  - Or, if Ford-Fulkerson terminates, then f is max flow.

[$A \Rightarrow B$]

- Weak duality corollary.

Max-flow min-cut theorem II

Max-flow min-cut theorem. [strong duality] Value of a max flow = capacity of a min cut.

Augmenting path theorem. A flow f is a max flow iff no augmenting paths.

Pf. The following three conditions are equivalent for any flow f:

- A. There exists a cut (A, B) such that $cap(A, B) = val(f)$.
- B. f is a max flow.
- C. There is no augmenting path with respect to f.

[$B \Rightarrow C$] We prove contrapositive: $\neg C \Rightarrow \neg B$.

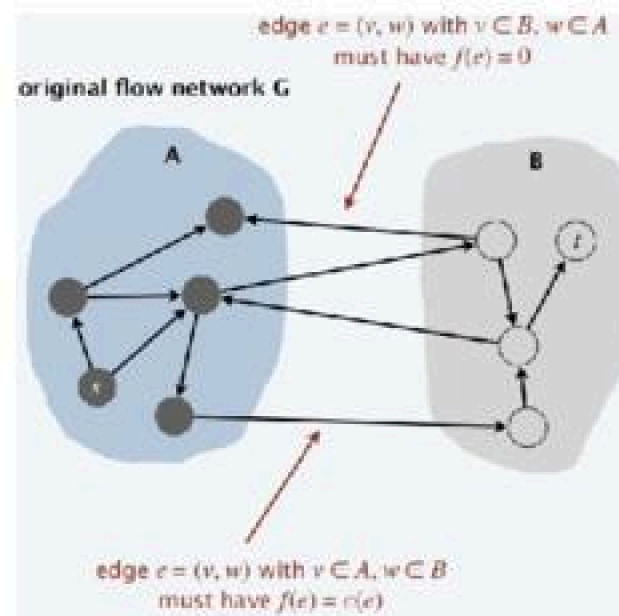
- Suppose that there is an augmenting path with respect to f.
- Can improve flow f by sending flow along this path.
- Thus, f is not a max flow, contradiction.

Max-flow min-cut theorem III

$[\, C \Rightarrow A \,]$

- Let f be a flow with no augmenting paths.
- Let A = set of nodes reachable from s in residual network G_f.
- By definition of A: $s \in A$.
- By definition of flow f: $t \notin A$.

$$
\begin{aligned}
val(f) &= \sum_{e \text{ out } A} f(e) - \sum_{e \text{ into } A} f(e) \\
&= \sum_{e \text{ out } A} c(e) - 0 \\
&= cap(A, B)
\end{aligned}
$$



Computing a minimum cut

Theorem. Given any max flow f, can compute a min cut (A, B) in $O(m)$ time.

Pf. Let A = set of nodes reachable from s in residual network G_f.

- argument from previous slide implies that capacity of (A, B) = value of flow f

Capacity-scaling algorithm

Ford-Fulkerson: analysis

Assumption. Every edge capacity $c(e)$ is an integer between 1 and C.

Integrality invariant. Throughout Ford-Fulkerson, every edge flow $f(e)$ and residual capacity $c_f(e)$ is an integer.
Pf. By induction on the number of augmenting paths.

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Theorem. Ford-Fulkerson terminates after at most $val(f^*) \leq nC$ augmenting paths, where f^* is a max flow.
Pf. Each augmentation increases the value of the flow by at least 1.

Corollary. The running time of Ford-Fulkerson is $O(mnC)$.
Pf. Can use either BFS or DFS to find an augmenting path in $O(m)$ time.

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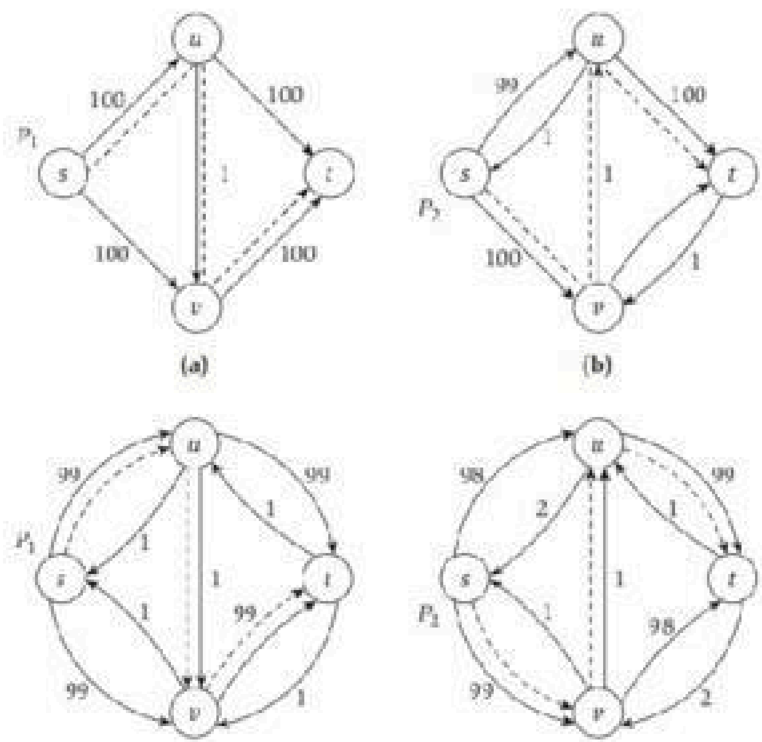
Pf. Can use either BFS or DFS to find an augmenting path in $O(m)$ time.

Integrality theorem. There exists an integral max flow f^*.
Pf. Since Ford-Fulkerson terminates, theorem follows from integrality invariant (and augmenting path theorem).

Ford-Fulkerson: exponential example

Q. Is generic Ford-Fulkerson algorithm poly-time in input size $(m, n, \log C)$?

A. No. If max capacity is C, then algorithm can take $\geq C$ iterations.



- See Demo.

Quiz: Ford-Fulkerson

The Ford-Fulkerson algorithm is guaranteed to terminate if the edge capacities are …

A. Rational numbers.
B. Real numbers.
C. Both A and B.
D. Neither A nor B.

Quiz: Ford-Fulkerson

The Ford-Fulkerson algorithm is guaranteed to terminate if the edge capacities are …

A. Rational numbers.
B. Real numbers.
C. Both A and B.
D. Neither A nor B.

Rational = integer / integer

Choosing good augmenting paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

Pathology. When edge capacities can be irrational, no guarantee that Ford-Fulkerson terminates (or converges to a maximum flow)!

- See Demo.

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Use care when selecting augmenting paths.

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- See Demo.

Goal. Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

Choosing good augmenting paths (cont.)

Goal. Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

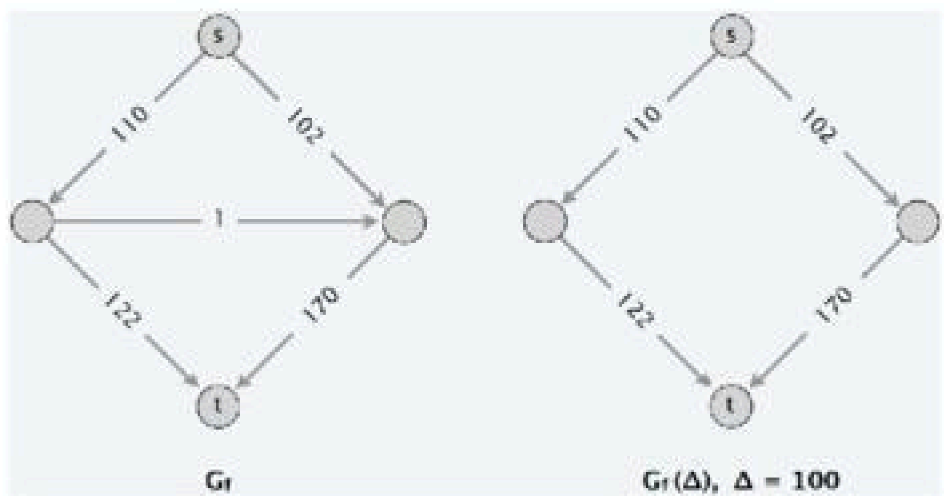
Choose augmenting paths with:

- Max bottleneck capacity ("fattest").
  - How to find?
  - [Next] Sufficiently large bottleneck capacity.
- [Ahead] Fewest edges.

Capacity-scaling

Overview. Choosing augmenting paths with "large" bottleneck capacity.

- Maintain scaling parameter Δ.
- Let $G_f(\Delta)$ be the part of the residual network containing only those edges with capacity $\geq \Delta$.
- Any augmenting path in $G_f(\Delta)$ has bottleneck capacity $\geq \Delta$.



Capacity-scaling: algorithm

CAPACITY-SCALING(G)

1. FOREACH edge $e \in E$: $f(e) = 0$;
2. Δ = largest power of $2 \leq C$;
3. WHILE ($\Delta \geq 1$)
   1. $G_f(\Delta)$ = Δ-residual network of G with respect to flow f;
   2. WHILE (there exists an $s \rightsquigarrow t$ path P in $G_f(\Delta)$)
      1. f = AUGMENT(f, c, P);
      2. Update $G_f(\Delta)$;
   3. $\Delta = \Delta / 2$;
4. RETURN f;

Capacity-scaling: correctness

Assumption. All edge capacities are integers between 1 and C.

Invariant. The scaling parameter Δ is a power of 2.
Pf. Initially a power of 2; each phase divides Δ by exactly 2.

Integrality invariant. Throughout the algorithm, every edge flow $f(e)$ and residual capacity $c_f(e)$ is an integer.
Pf. Same as for generic Ford-Fulkerson.

Capacity-scaling: correctness

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Invariant. The scaling parameter Δ is a power of 2.
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Integrality invariant. Throughout the algorithm, every edge flow $f(e)$ and residual capacity $c_f(e)$ is an integer.
Pf. Same as for generic Ford-Fulkerson.

Theorem. If capacity-scaling algorithm terminates, then f is a max flow.
Pf.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths.
- Result follows augmenting path theorem.

Capacity-scaling: analysis

Lemma 1. There are $1 + \lfloor \log_2 C \rfloor$ scaling phases.
Pf. Initially $C/2 < \Delta \leq C$; Δ decreases by a factor of 2 in each iteration.

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then, the max-flow value $\leq val(f) + m\Delta$.
Pf. Next slide.

Lemma 3. There are $\leq 2m$ augmentations per scaling phase.
Pf.

- Let f be the flow at the beginning of a Δ-scaling phase.
- Lemma 2 $\Rightarrow$ max-flow value $\leq val(f) + m(2\Delta)$.
- Each augmentation in a Δ-phase increases $val(f)$ by at least Δ.

≡

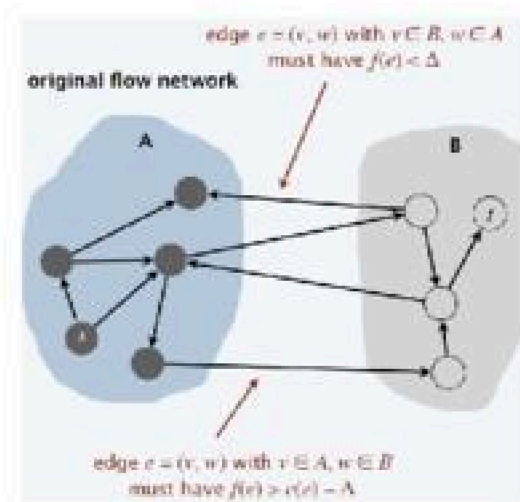
Capacity-scaling: analysis (cont.)

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then, the max-flow value $\leq val(f) + m\Delta$.

Pf.

- We show there exists a cut (A, B) such that $cap(A, B) \leq val(f) + m\Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of A: $s \in A$; By definition of flow f: $t \notin A$.

$$
\begin{aligned}
val(f) &= \sum_{e \text{ out } A} f(e) - \sum_{e \text{ into } A} f(e) \\
&\geq \sum_{e \text{ out } A} (c(e) - \Delta) - \sum_{e \text{ into } A} \Delta \\
&\geq \sum_{e \text{ out } A} c(e) - \sum_{e \text{ out } A} \Delta - \sum_{e \text{ into } A} \Delta \\
&\geq cap(A, B) m\Delta
\end{aligned}
$$



Capacity-scaling: running time

Lemma 1. There are $1 + \lfloor \log_2 C \rfloor$ scaling phases.

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then, the max-flow value $\leq val(f) + m\Delta$.

Lemma 3. There are $\leq 2m$ augmentations per scaling phase.

Theorem. The capacity-scaling algorithm takes $O(m^2 \log C)$ time.
Pf.

- Lemma 1 + Lemma 3 $\Rightarrow O(mlogC)$ augmentations.
- Finding an augmenting path takes $O(m)$ time.

Shortest augmenting paths

Shortest augmenting

Q. How to choose next augmenting path in Ford-Fulkerson?
A. Pick one that uses the fewest edges (*via BFS*).

SHORTEST-AUGMENTING-PATH(G)

1. FOREACH $e \in E$: $f(e) = 0$;
2. G_f = residual network of G with respect to flow f;
3. WHILE (there exists an $s \rightsquigarrow t$ path in G_f)
   1. P = BREADTH-FIRST-SEARCH(G_f);
   2. f = AUGMENT(f, c, P);
   3. Update G_f;
4. RETURN f;

Shortest augmenting: analysis overview

Lemma 1. The length (number of edges) of a shortest augmenting path never decreases.
Pf. Ahead.

Lemma 2. After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases.
Pf. Ahead.

Shortest augmenting: analysis overview

Lemma 1. The length (number of edges) of a shortest augmenting path never decreases.
Pf. Ahead.

Lemma 2. After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases.
Pf. Ahead.

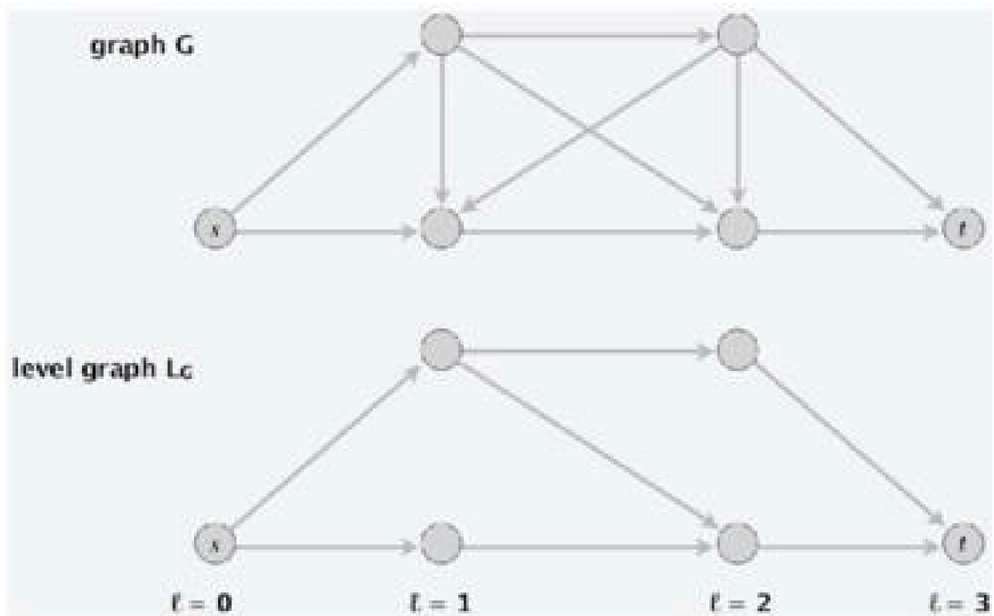
Theorem. The shortest-augmenting-path algorithm takes $O(m^2 n)$ time.
Pf.

- $O(m)$ time to find a shortest augmenting path via BFS.
- There are $\leq mn$ augmentations.
  - [from Lemmas] at most m augmenting paths of length k
  - [simple path] at most $n - 1$ different lengths

Level graph

Def. Given a digraph $G = (V, E)$ with source s, its **level graph** is defined by:

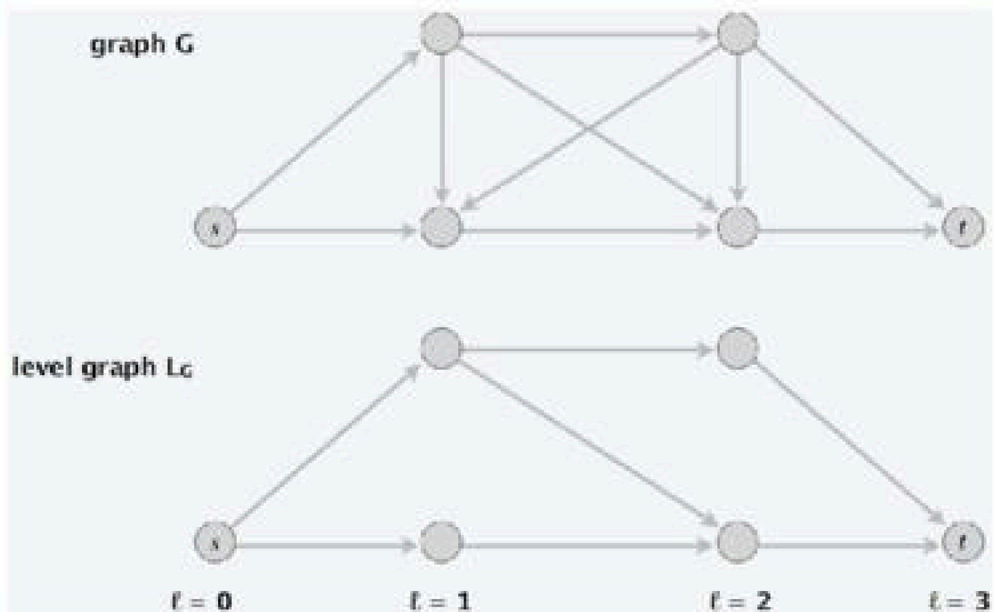
- l(v) = number of edges in shortest $s \rightsquigarrow v$ path.
- $L_G = (V, E_G)$ is the subgraph of G that contains only those edges $(v, w) \in E$ with $l(w) = l(v) + 1$.



Level graph (cont.)

Key property. P is a shortest $s \rightsquigarrow v$ path in G iff P is an $s \rightsquigarrow v$ path in L_G.

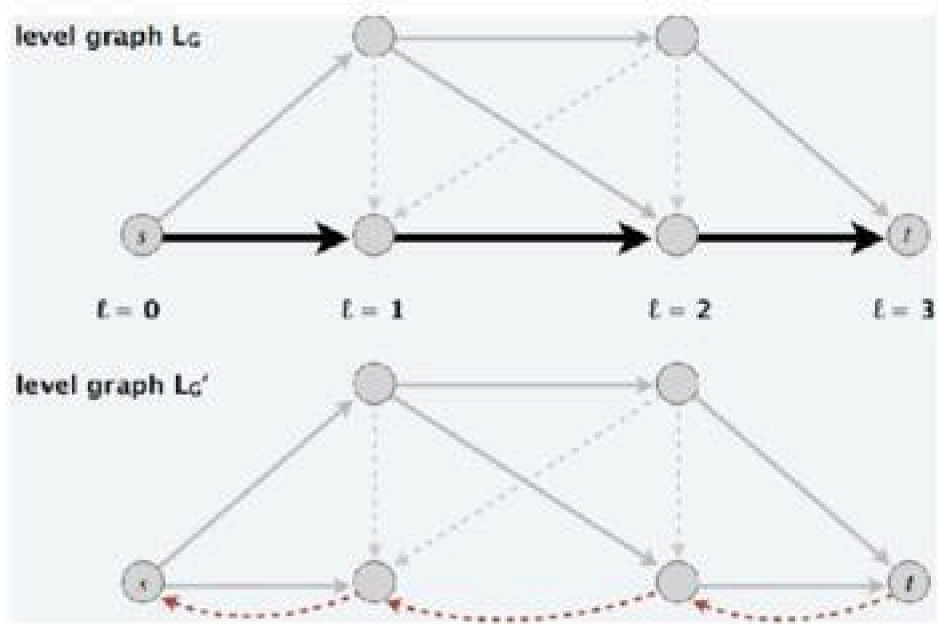
- nodes are ordered the same with BFS
- "back-edges" are removed



Shortest augmenting: Lemma 1

Lemma 1. The length of a shortest augmenting path never decreases.

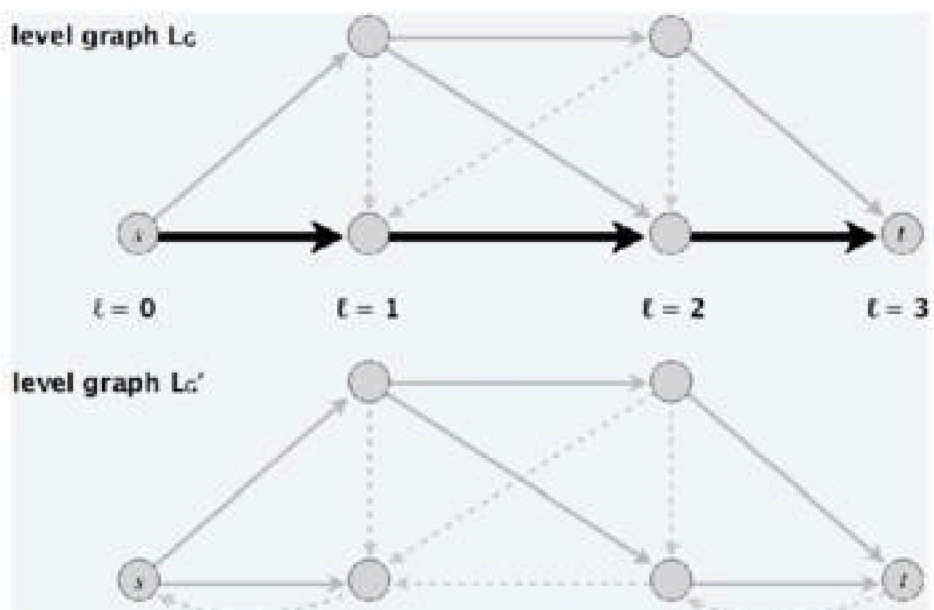
- Let f and f' be flow before and after a shortest-path augmentation.
- Let L_G and $L_G{}'$ be level graphs of G_f and G'_f.
- Only back edges added to G'_f; bottleneck broken.
- any $s \rightsquigarrow t$ path uses back edge is longer than previous length.



Shortest augmenting: Lemma 2

Lemma 2. After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases.

- At least one (bottleneck) edge is deleted from L_G per augmentation.
- No new edge added to L_G until shortest path length strictly increases.



Shortest augmenting: analysis

Lemma 1. The length (number of edges) of a shortest augmenting path never decreases.

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Theorem. The shortest-augmenting-path algorithm takes $O(m^2 n)$ time.

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Note. $\Theta(mn)$ augmentations necessary for some flow networks.

- Try to decrease time per augmentation instead.
  - Simple idea $\Rightarrow O(mn^2)$ [Dinitz 1970]
  - Dynamic trees $\Rightarrow O(mn \log n)$ [Sleator-Tarjan 1983]

Dinitz' algorithm

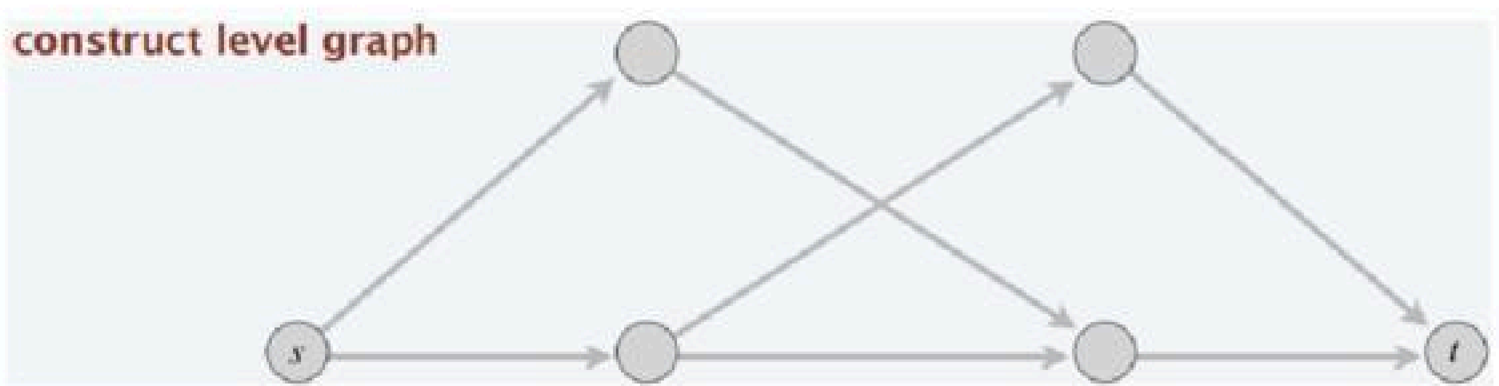
Dinitz' algorithm

Two types of augmentations.

- Normal: length of shortest path does not change.
- Special: length of shortest path strictly increases.

Phase of normal augmentations.

- Construct level graph L_G.
- Start at s, advance along an edge in L_G until reach t or get stuck.
  - If reach t, augment flow; update L_G; and restart from s.
  - If get stuck, delete node from L_G and retreat to previous node.



Demo: Dinitz' algorithm

Dinitz' algorithm (refined)

INITIALIZE(G, f)

1. L_G = level-graph of G_f;
2. $P = \emptyset$;
3. GOTO ADVANCE(s);

RETREAT(v)

1. IF $(v = s)$ STOP;
2. ELSE
   1. Delete v (and all incident edges) from L_G;
   2. Remove last edge (u, v) from P ;
   3. GOTO ADVANCE(u);

ADVANCE(v)

1. IF $(v = t)$ AUGMENT(P);
   1. Remove saturated edges from L_G;
   2. $P = \emptyset$;
   3. GOTO ADVANCE(s);
2. IF (there exists edge $(v, w) \in L_G$)
   1. Add edge (v, w) to P;
   2. GOTO ADVANCE(w);
3. ELSE
   1. GOTO RETREAT(v);

Quiz: level graph

How to compute the level graph L_G efficiently?

A. Depth-first search.
B. Breadth-first search.
C. Both A and B.
D. Neither A nor B.

Dinitz' algorithm: analysis

Lemma. A phase can be implemented to run in $O(mn)$ time.

Pf.

- Initialization happens once per phase. $O(m)$ using BFS.
- At most m augmentations per phase. $O(mn)$ per phase.
  - (because an augmentation deletes at least one edge from L_G)
- At most n retreats per phase. $O(m + n)$ per phase
  - (because a retreat deletes one node from L_G)
- At most mn advances per phase. O(mn) per phase
  - (because at most n advances before retreat or augmentation)

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Theorem. [Dinitz 1970] Dinitz' algorithm runs in $O(mn^2)$ time.

Pf.

- By Lemma, $O(mn)$ time per phase.
- At most $n-1$ phases (as in shortest-augmenting-path analysis).

Maximum-flow algorithms: practice

Push-relabel algorithm. [Goldberg-Tarjan 1988] Increases flow one edge at a time instead of one augmenting path at a time.

- (SECTION 7.4, Algorithm Design.)

Maximum-flow algorithms: practice

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Caveat. Worst-case running time is generally not useful for predicting or comparing max-flow algorithm performance in practice.

Best in practice. Push-relabel method with gap relabeling: $O(m^{3/2})$ in practice.

Maximum-flow algorithms: CV

Computer vision. Different algorithms work better for some dense problems that arise in applications to computer vision.

- [Boykov and Kolmogorov 2004] An experimental comparison of min-cut/max- flow algorithms for energy minimization in vision.

Simple unit-capacity networks

Quiz: bipartite matching

Which max-flow algorithm to use for bipartite matching?

A. Ford-Fulkerson: $O(mnC)$.
B. Capacity scaling: $O(m^2 logC)$.
C. Shortest augmenting path: $O(m^2 n)$.
D. Dinitz' algorithm: $O(mn^2)$.

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D. Dinitz' algorithm: $O(mn^2)$.

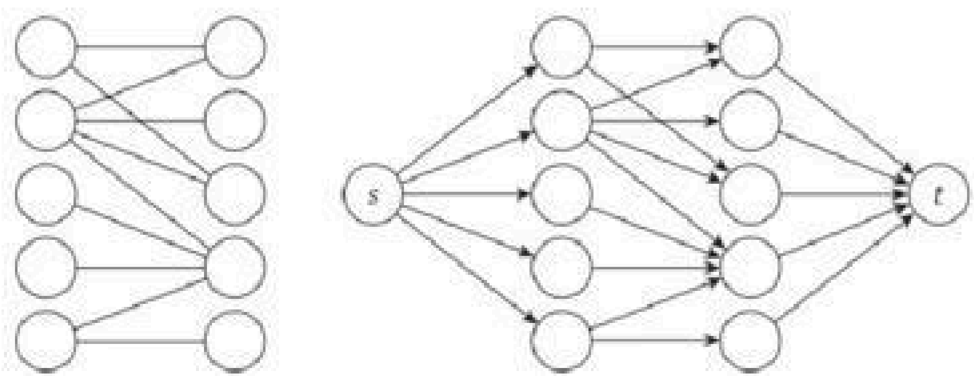
D. the graph may be dense.

Simple unit-capacity networks

Def. A flow network is a **simple unit-capacity network** if:

- Every edge has capacity 1.
- Every node (other than s or t) has exactly one entering edge, or exactly one leaving edge, or both.

Ex. Bipartite matching.



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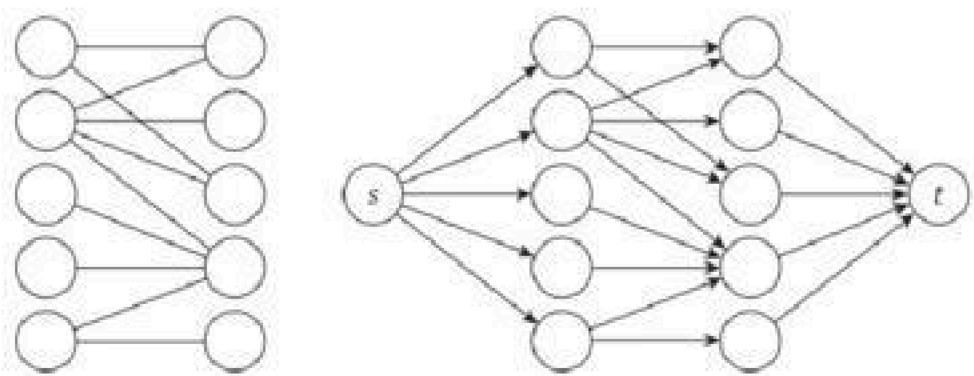

Property. Let G be a simple unit-capacity network and let f be a 0-1 flow. Then, residual network G_f is also a simple unit-capacity network.

Unit-capacity: algorithm overview

Shortest-augmenting-path algorithm.

- Normal augmentation: length of shortest path does not change.
- Special augmentation: length of shortest path strictly increases.

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Theorem. [Even-Tarjan 1975] In simple unit-capacity networks, Dinitz' algorithm computes a maximum flow in $O(mn^{1/2})$ time.

Pf.

- Lemma 1. Each phase of normal augmentations takes $O(m)$ time.
- Lemma 2. After $n^{1/2}$ phases, $val(f) \geq val(f^*) - n^{1/2}$.
- Lemma 3. After $\leq n^{1/2}$ additional augmentations, flow is optimal.

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Pf. Each augmentation increases flow value by at least 1.

Lemma 1 and Lemma 2. Ahead.

Unit-capacity: Dinitz'

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- Construct level graph L_G.
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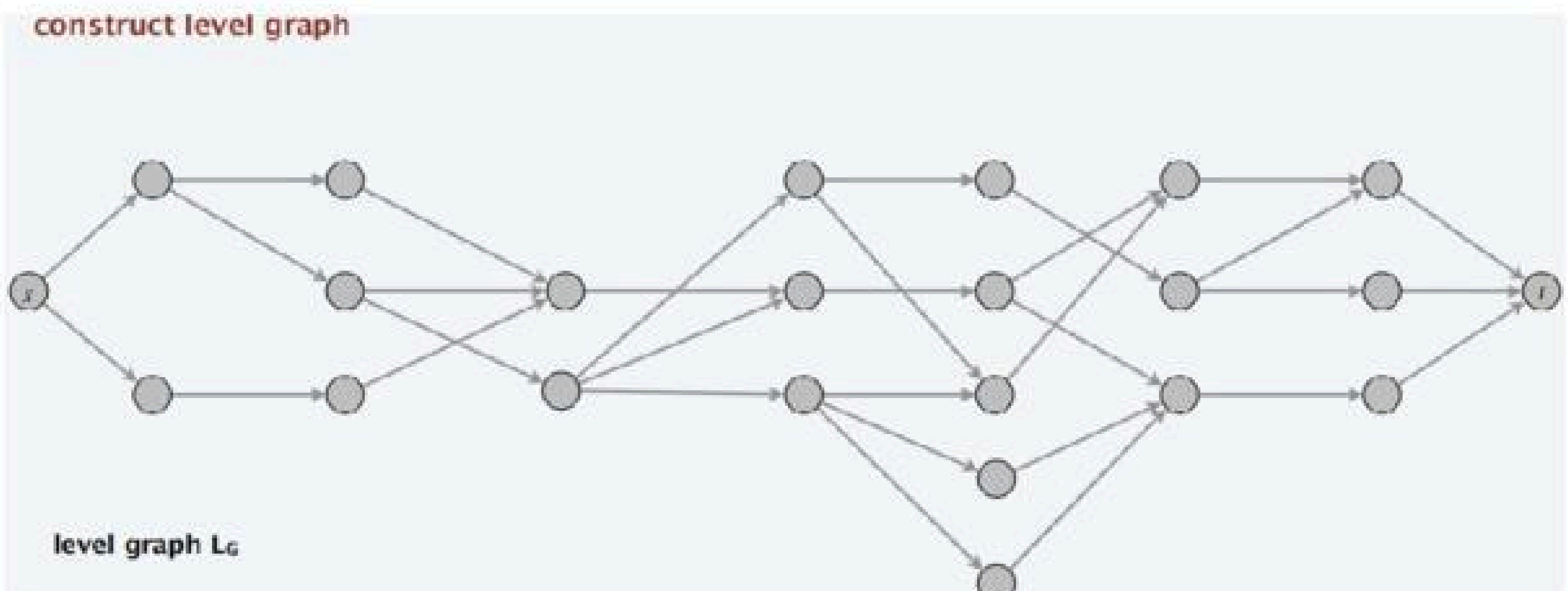

Demo: Dinitz' for Unit-capacity

Unit-capacity: Lemma 1

Phase of normal augmentations.

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- If reach t, augment flow; update L_G; and restart from s.
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Lemma 1. A phase of normal augmentations takes $O(m)$ time.
Pf.

- $O(m)$ to create level graph L_G.
- $O(1)$ per edge (each edge involved in at most one advance, retreat, and augmentation).
- $O(1)$ per node (each node deleted at most once).

Quiz: non-unit-capacity

Consider running advance-retreat algorithm in a unit-capacity network (but not necessarily a simple one). What is running time?

A. $O(m)$.
B. $O(m^{3/2})$.
C. $O(mn)$.
D. May not terminate.

Hint: both indegree and outdegree of a node can be larger than 1.

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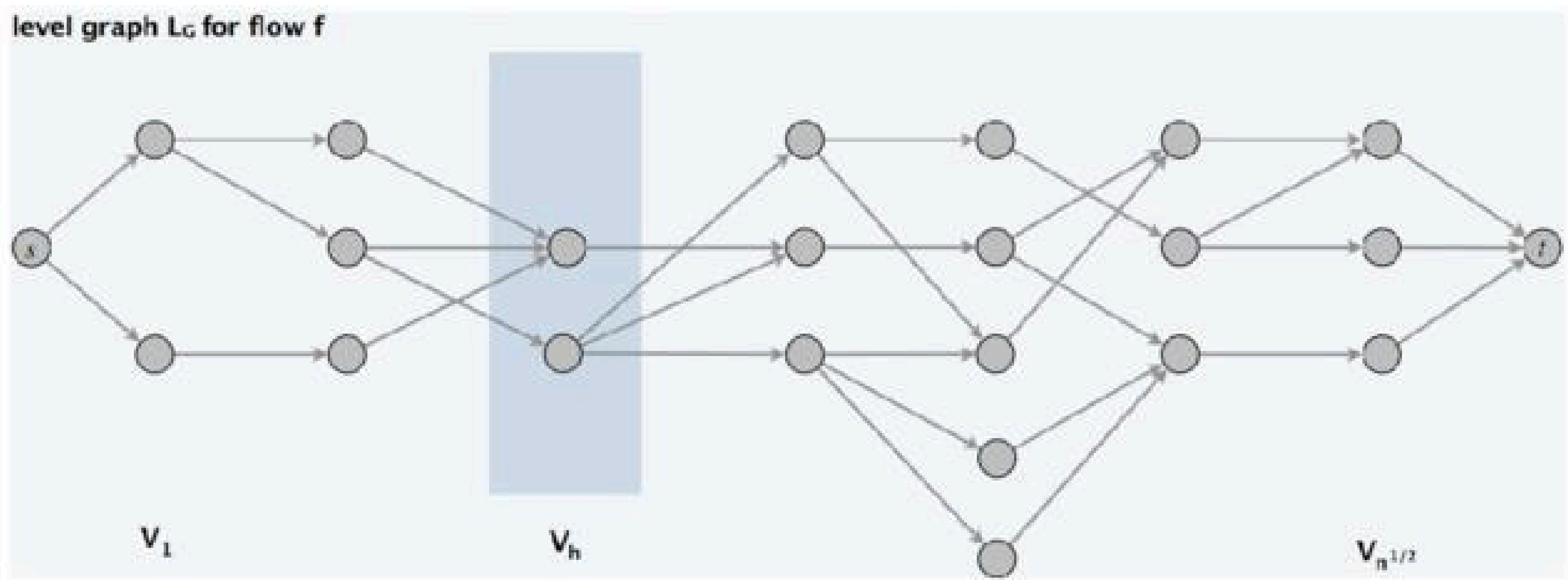
Hint: both indegree and outdegree of a node can be larger than 1.

A. may take m operations per node

Unit-capacity: Lemma 2

Lemma 2. After $n^{1/2}$ phases, $val(f) \geq val(f^*) - n^{1/2}$.

- After $n^{1/2}$ phases, length of shortest augmenting path is $> n^{1/2}$.
- Thus, level graph has $\geq n^{1/2}$ levels (not including levels for s or t).
- Let $1 \leq h \leq n^{1/2}$ be a level with min number of nodes $\Rightarrow |V_h| \leq n^{1/2}$.



Unit-capacity: Lemma 2 (cont.)

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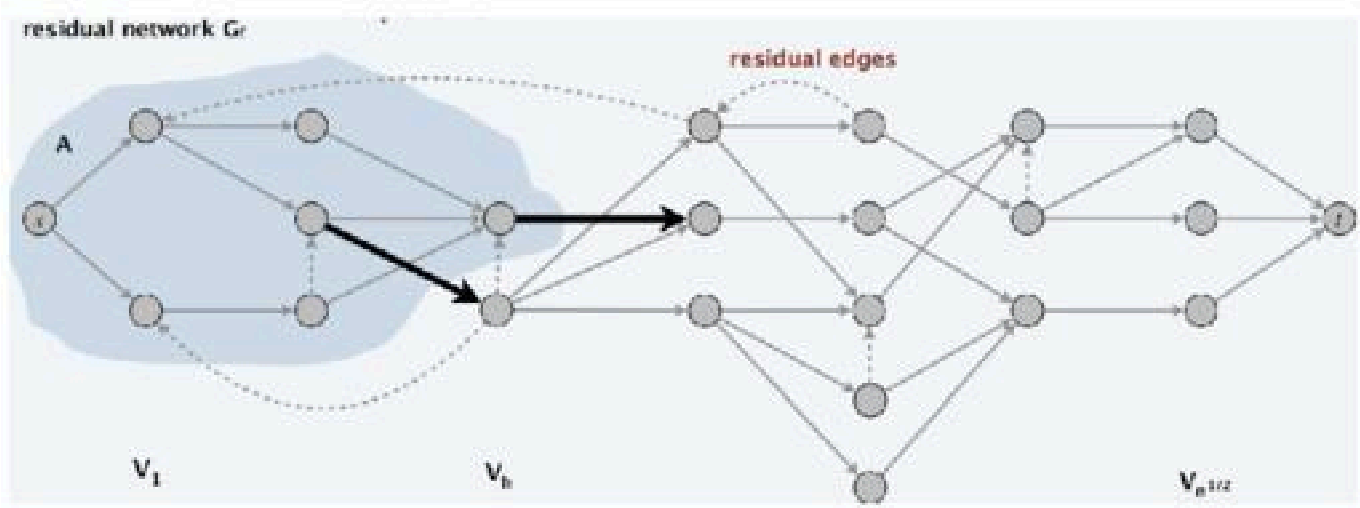
- Let $1 \leq h \leq n^{1/2}$ be a level with min number of nodes $\Rightarrow |V_h| \leq n^{1/2}$.
- Let $A = \{v : l(v) < h\} \cup \{v : l(v) = h \text{ and } v \text{ has } \leq 1 \text{ outgoing residual edge}\}$.
- $cap_f(A, B) \leq |V_h| \leq n^{1/2} \Rightarrow val(f) \geq val(f^*) - n^{1/2}$.



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Corollary. Dinitz' algorithm computes max-cardinality bipartite matching in $O(mn^{1/2})$ time.