Algorithm II

6. Dynamic Programming II

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Sequence alignment

String similarity

Q. How similar are two strings?

Ex. ocurrance and occurrence.

o	c	u	r	r	a	n	c	e	-	
o	c	c	u	r	r	e	n	c	e	
		*	*		*	*	*	*	-	
o	c	-	u	r	r	a	n	c	e	
o	c	c	u	r	r	e	n	c	e	
		-				*				
o	c	-	u	r	r	-	a	n	c	e
o	c	c	u	r	r	e	-	n	c	e
		-				-	-			

Edit distance

Edit distance. [Levenshtein 1966, Needleman–Wunsch 1970]

- **Gap** penalty δ; **mismatch** penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.

C	T	–	G	A	C	C	T	A	C	G
C	T	G	G	A	C	G	A	A	C	G
		-				-	-			

- cost = $\delta + \alpha_{CG} + \alpha_{TA}$.

Applications. Bioinformatics, spell correction, machine translation, speech recognition, information extraction, etc.

Sequence alignment: cost

Goal. Given two strings $x_1 x_2 ... x_m$ and $y_1 y_2 ... y_n$, find a min-cost alignment.

Def. An alignment M is a set of ordered pairs $x_i – y_j$ such that each character appears in at most one pair and no crossings.

- $x_i – y_j$ and $x_{i'} – y_{j'}$ cross if $i < i'$, but $j > j'$.

Def. The cost of an alignment M is:

- $cost(M) = \sum_{(x_i, y_j) \in M} \alpha_{x_i y_j} + \sum_{i : x_i \, unmatched} \delta + \sum_{j : y_j \, unmatched} \delta$

Sequence alignment: problem structure

Def. $OPT(i, j)$ = min cost of aligning prefix strings $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$.

Goal. $OPT(m, n)$.

Case 1. $OPT(i, j)$ matches $x_i – y_i$.

- Pay mismatch for $x_i – y_i$ + min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_{i-1}$.

Case 2a. $OPT(i, j)$ leaves x_i unmatched.

- Pay gap for x_i + min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_i$.

Case 2b. $OPT(i, j)$ leaves y_i unmatched.

- Pay gap for y_i + min cost of aligning $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{i-1}$.

≡

Sequence alignment: Bellman equation

Def. $OPT(i, j)$ = min cost of aligning prefix strings $x_1 x_2 ... x_i$ and $y_1 y_2 ... y_j$.

Goal. $OPT(m, n)$.

Case 1. $OPT(i, j)$ matches $x_i – y_i$.

Case 2a. $OPT(i, j)$ leaves x_i unmatched.

Case 2b. $OPT(i, j)$ leaves y_i unmatched.

Bellman equation.

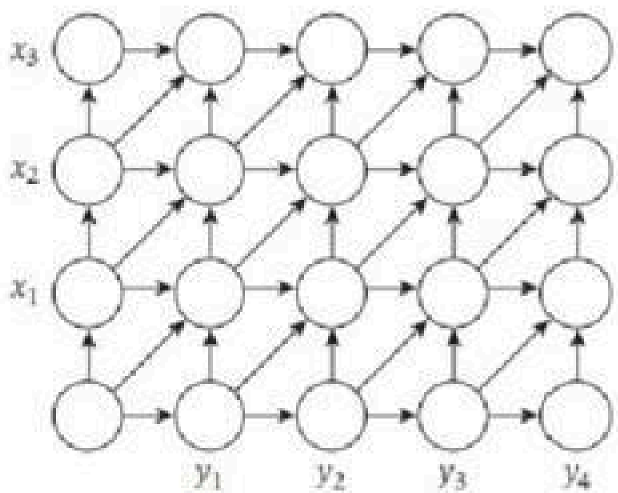
$$OPT(i,j) = \begin{cases} j\delta \\ i\delta \\ \min\{\alpha_{x_i,y_j} + OPT(i-1, j-1), \delta + OPT(i-1, j), \delta + OPT(i, j-1)\} \end{cases}$$

≡

Sequence alignment: Algorithm

SEQUENCE-ALIGNMENT$(m, n, x_1, \ldots, x_m, y_1, \ldots, y_n, \delta, \alpha)$

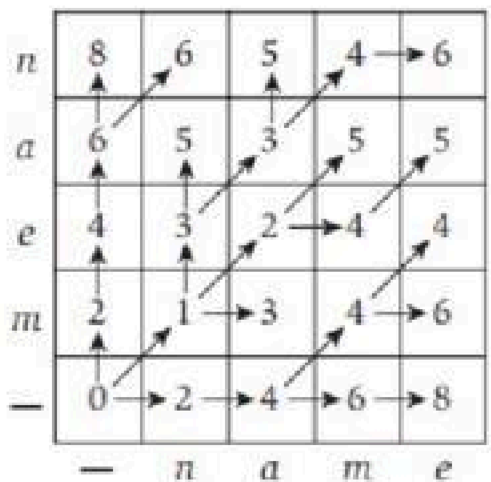
1. FOR $i = 0..m$: $M[i, 0] = i\delta$;
2. FOR $j = 0..n$: $M[0, j] = j\delta$;
3. FOR $i = 1..m$:
   1. FOR $j = 1..n$:
      1. $M[i, j] = \min\{\alpha x_i y_j + M[i-1, j-1], \delta + M[i-1, j], \delta + M[i, j-1]\}$;
4. RETURN $M[m, n]$;



Sequence alignment: trace-back

Ex. Matching *mean* and *name*, with:

- $\delta = 2$,
- mismatch vowels or consonants cost 1.
- matching vowel and consonant cost 3.



m	e	a	n	-
n	-	a	m	e
*	-		*	-
1	2	0	1	2

Sequence alignment: analysis

Theorem. The DP algorithm computes the edit distance (and an optimal alignment) of two strings of lengths m and n in $\Theta(mn)$ time and space.
Pf.

Correctness.

- Algorithm computes edit distance.
- Can trace back to extract optimal alignment itself.

Time. M has mn entries.

Hirschberg's algorithm

Sequence alignment in linear space

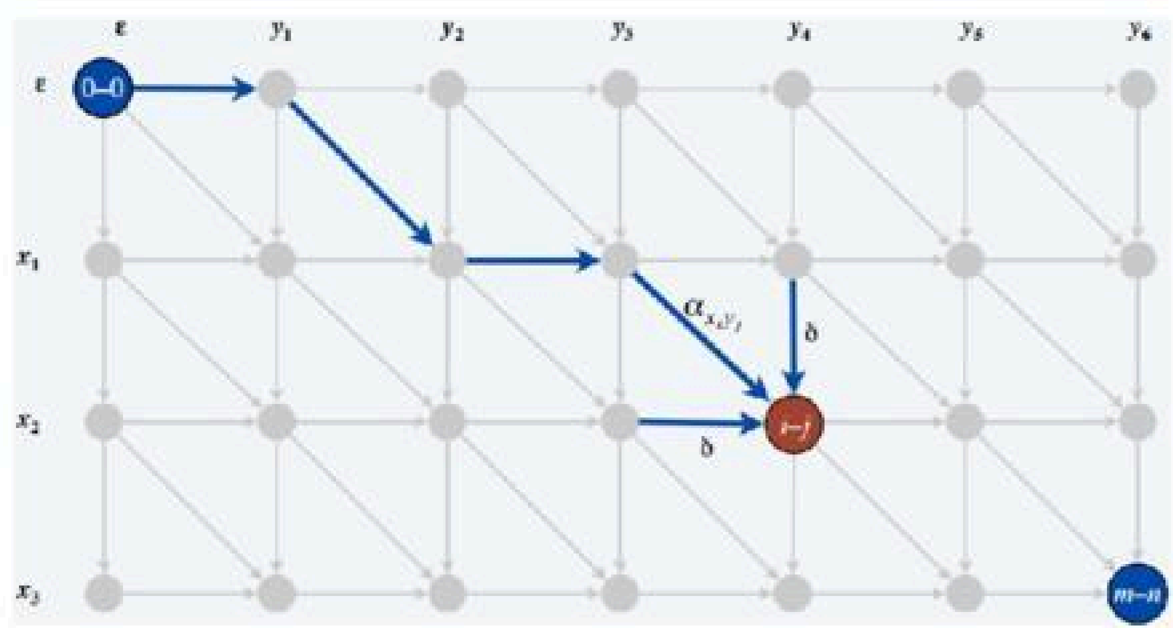
Theorem. [Hirschberg] There exists an algorithm to find an optimal alignment in $O(mn)$ time and $O(m + n)$ space.

- Clever combination of divide-and-conquer and dynamic programming.

Hirschberg's algorithm 1

Edit distance graph.

- Let $f(i, j)$ denote length of shortest path from $(0, 0)$ to (i, j).
- **Lemma**: $f(i, j) = OPT(i, j)$ for all i and j.



Hirschberg's algorithm 1.1

Edit distance graph.

- Let $f(i, j)$ denote length of shortest path from $(0, 0)$ to (i, j).
- **Lemma**: $f(i, j) = OPT(i, j)$ for all i and j.

Pf of Lemma. [by strong induction on $i + j$]

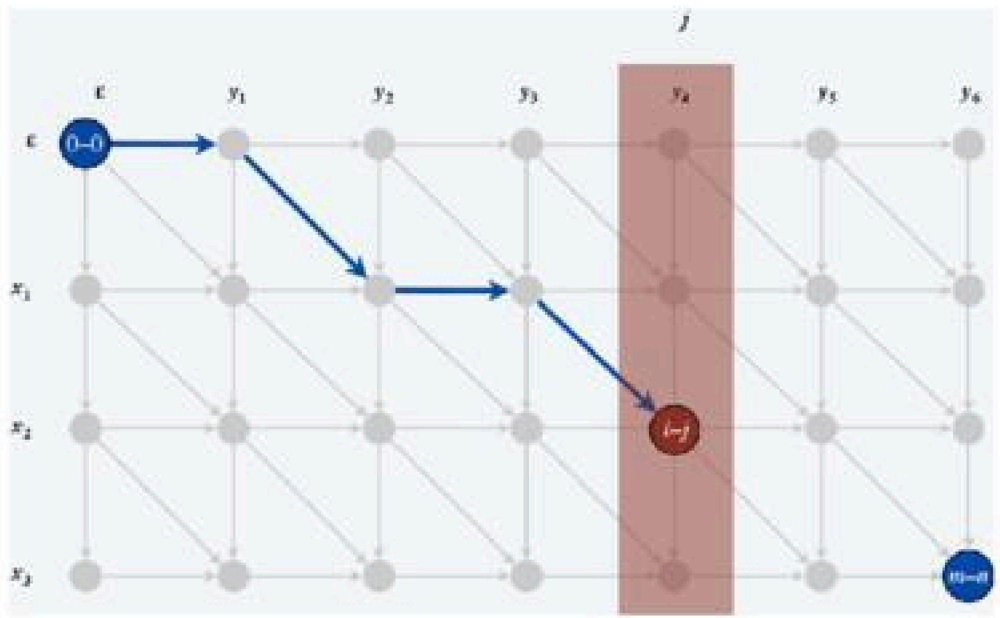
- Base case: $f(0, 0) = OPT(0, 0) = 0$.
- Inductive hypothesis: assume true for all (i', j') with $i' + j' < i + j$.
- Last edge on shortest path to (i, j) is from $(i–1, j–1)$, $(i–1, j)$, or $(i, j–1)$.
- Thus,

$$
\begin{aligned}
f(i,j) &= \min\{\alpha_{x_i,y_j} + f(i-1,j-1), \delta + f(i-1,j), \delta + f(i,j-1)\} \\
&= \min\{\alpha_{x_i,y_j} + OPT(i-1,j-1), \delta + OPT(i-1,j), \delta + OPT(i,j-1)\} \\
&= OPT(i,j)
\end{aligned}
$$

Hirschberg's algorithm 1.2

Edit distance graph.

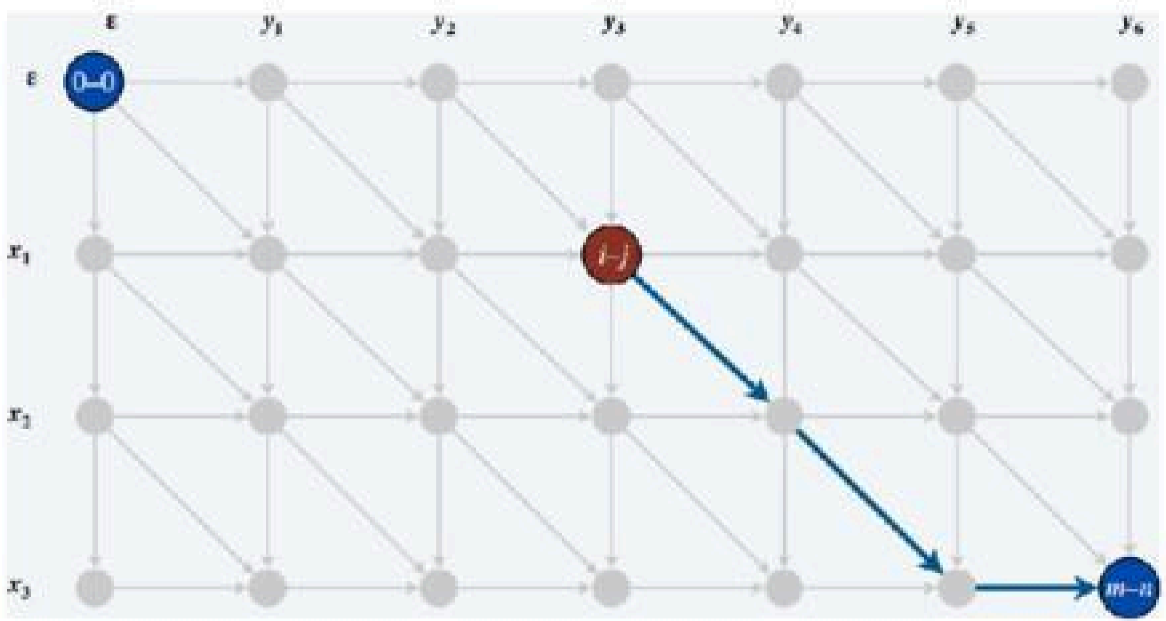
- Let $f(i, j)$ denote length of shortest path from $(0, 0)$ to (i, j).
- **Lemma**: $f(i, j) = OPT(i, j)$ for all i and j.
- Can compute $f(\cdot, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



Hirschberg's algorithm 2

Edit distance graph.

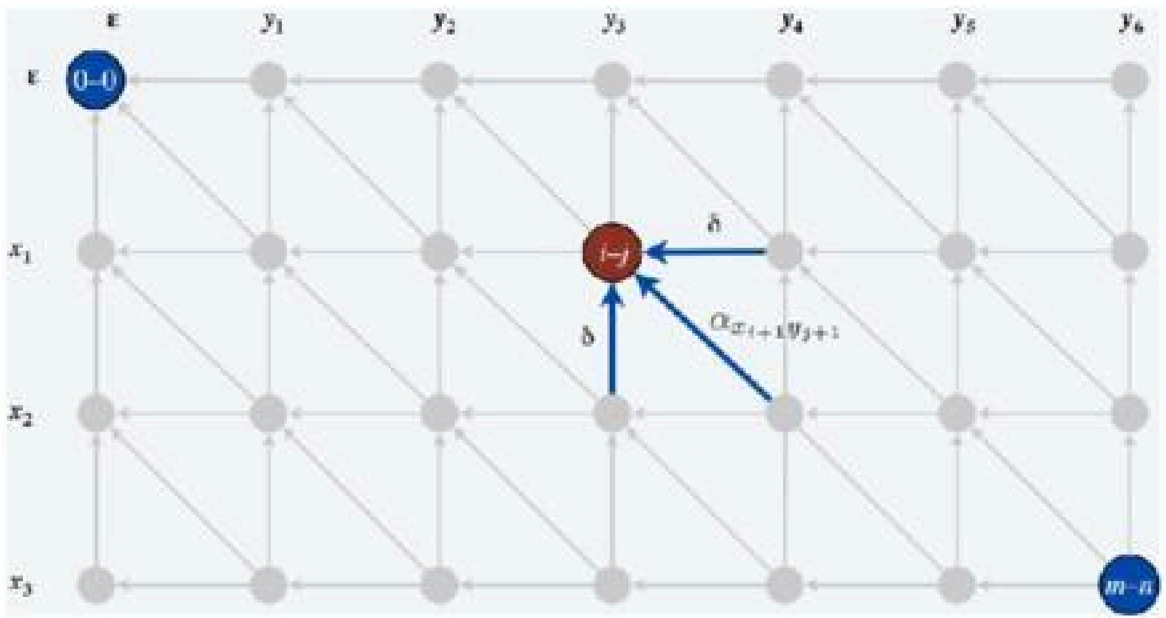
- Let $g(i, j)$ denote length of shortest path from (i, j) to (m, n).



Hirschberg's algorithm 2.1

Edit distance graph.

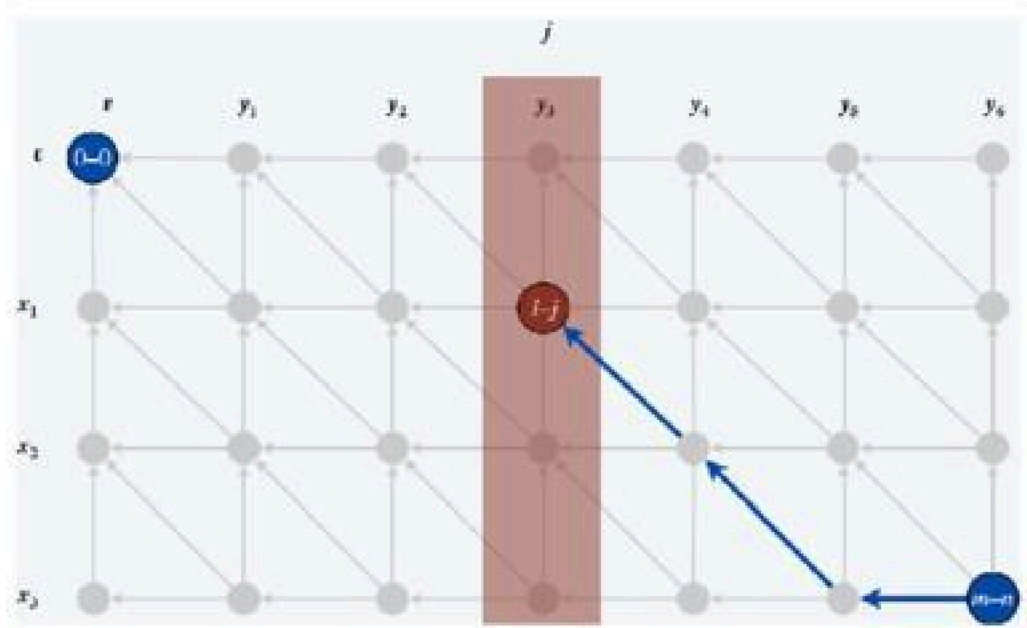
- Let $g(i, j)$ denote length of shortest path from (i, j) to (m, n).
- Can compute $g(i, j)$ by reversing the edge orientations and inverting the roles of $(0, 0)$ and (m, n).



Hirschberg's algorithm 2.2

Edit distance graph.

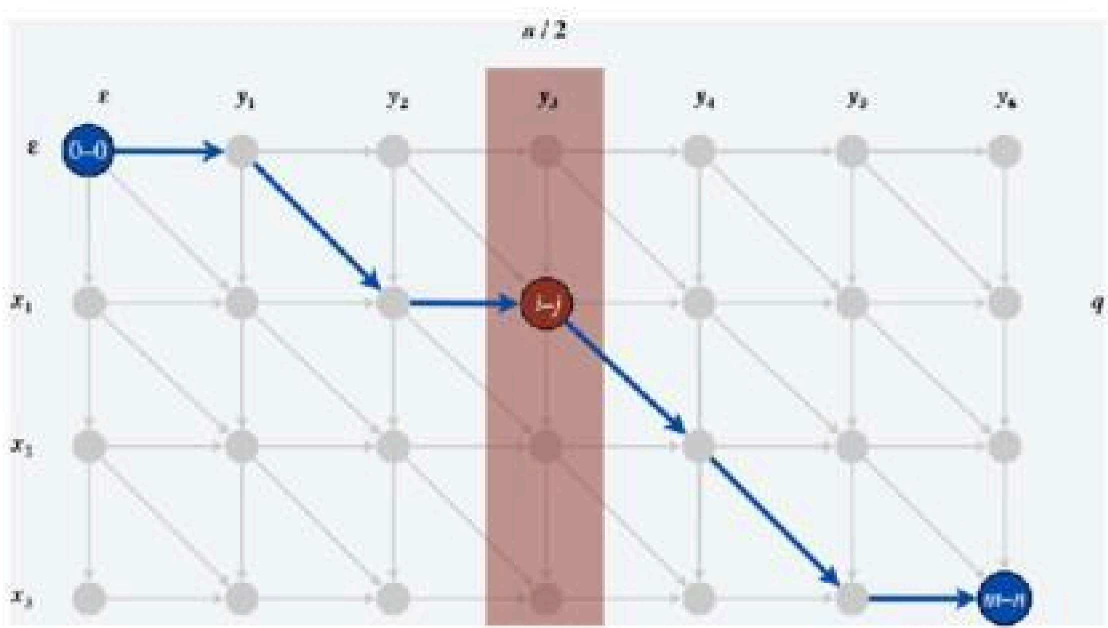
- Let $g(i, j)$ denote length of shortest path from (i, j) to (m, n).
- Can compute $g(\cdot, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



Hirschberg's algorithm 3

Observation 1. The length of a shortest path that uses (i, j) is $f(i, j) + g(i, j)$.

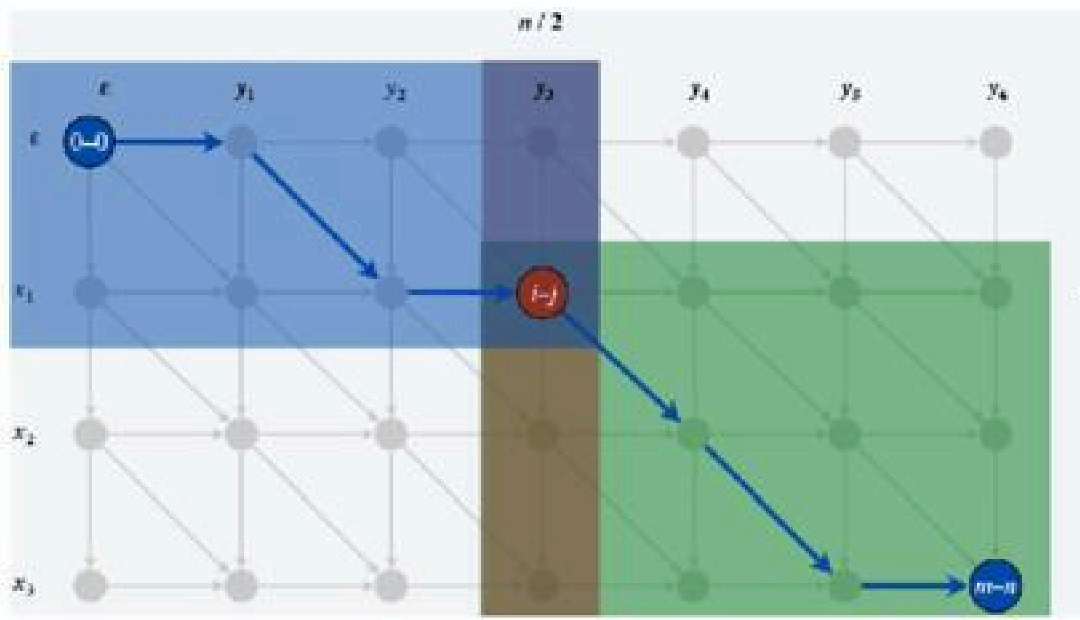
Observation 2. let q be an index that minimizes $f(q, n/2) + g(q, n/2)$. Then, there exists a shortest path from $(0, 0)$ to (m, n) that uses $(q, n/2)$.



Hirschberg's algorithm 4

Divide. Find index q that minimizes $f(q, n/2) + g(q, n/2)$; save node i–j as part of solution.

Conquer. Recursively compute optimal alignment in each piece.



Hirschberg's: space analysis

Theorem. Hirschberg's algorithm uses $\Theta(m+n)$ space.

Pf.

- Each recursive call uses $\Theta(m)$ space to compute $f(\cdot, n/2)$ and $g(\cdot, n/2)$.
- Only $\Theta(1)$ space needs to be maintained per recursive call.
- Number of recursive calls $\leq n$.

Hirschberg's: time analysis warmup

Theorem. Let $T(m, n)$ = max running time of Hirschberg's algorithm on strings of lengths at most m and n. Then, $T(m, n) = O(mn \log n)$.

Pf.

- $T(m, n)$ is monotone non-decreasing in both m and n.
- $T(m, n) \leq 2T(m, n/2) + O(mn)$
  - $\Rightarrow T(m, n) = O(mn \log n)$.

Remark. Analysis is not tight because two sub-problems are of size $(q, n/2)$ and $(m-q, n/2)$. Next, we prove $T(m, n) = O(mn)$.

Hirschberg's: time analysis

Theorem. Let $T(m, n)$ = max running time of Hirschberg's algorithm on strings of lengths at most m and n. Then, $T(m, n) = O(mn)$.

Pf. [by strong induction on $m + n$]

- $O(mn)$ time to compute $f(\cdot, n/2)$ and $g(\cdot, n/2)$ and find index q.
- $T(q, n/2) + T(m\text{–}q, n/2)$ time for two recursive calls.
- For some constant c:

$$T(m, n) \leq cmn + T(q, n/2) + T(m - q, n/2)$$
$$T(m, 2) \leq cm$$
$$T(2, n) \leq cn$$

Claim. $T(m, n) \leq 2cmn$.

Hirschberg's: time analysis (cont.)

$$T(m,n) \le cmn + T(q, n/2) + T(m-q, n/2)$$
$$T(m,2) \le cm$$
$$T(2,n) \le cn$$

Claim. $T(m,n) \le 2cmn$.
Pf. [by strong induction on $m+n$]

- Base cases: $m = 2$ and $n = 2$.

$$\begin{aligned} T(m,n) &\le T(q, n/2) + T(m-q, n/2) + cmn \\ &\le 2cqn/2 + 2c(m-q)n/2 + cmn \\ &= cqn + cmn - cqn + cmn \\ &= 2cmn \end{aligned}$$

≡

Longest Common Subsequence

Problem. Given two strings $x_1 x_2 ... x_m$ and $y_1 y_2 ... y_n$, find a common subsequence that is as long as possible.

Alternative viewpoint. Delete some characters from x; delete some character from y; a common subsequence if it results in the same string.

Ex. LCS(GGCACCACG, ACGGCGGATACG) = GGCAACG.

Applications. Unix diff, git, bioinformatics.

Longest Common Subsequence: DP

Def. $OPT(i, j)$ = length of LCS of prefix strings $x_1 x_2 ... x_i$ and $y_1 y_2 ... y_j$.

Goal. $OPT(m, n)$.

Case 1. $x_i = y_j$.

- 1 + length of LCS of $x_1 x_2 ... x_{i-1}$ and $y_1 y_2 ... y_{j-1}$.

Case 2. $x_i \neq y_j$.

- Delete x_i: length of LCS of $x_1 x_2 ... x_{i-1}$ and $y_1 y_2 ... y_j$.
- Delete y_j: length of LCS of $x_1 x_2 ... x_i$ and $y_1 y_2 ... y_{j-1}$.

Bellman equation.

$$OPT(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ 1 + OPT(i-1, j-1) & \text{if } x_i = y_j \\ \max\{OPT(i-1, j), OPT(i, j-1)\} & \text{if } x_i \neq y_j \end{cases}$$

Longest Common Subsequence: DP II

Solution 2. Reduce to finding a min-cost alignment of x and y with

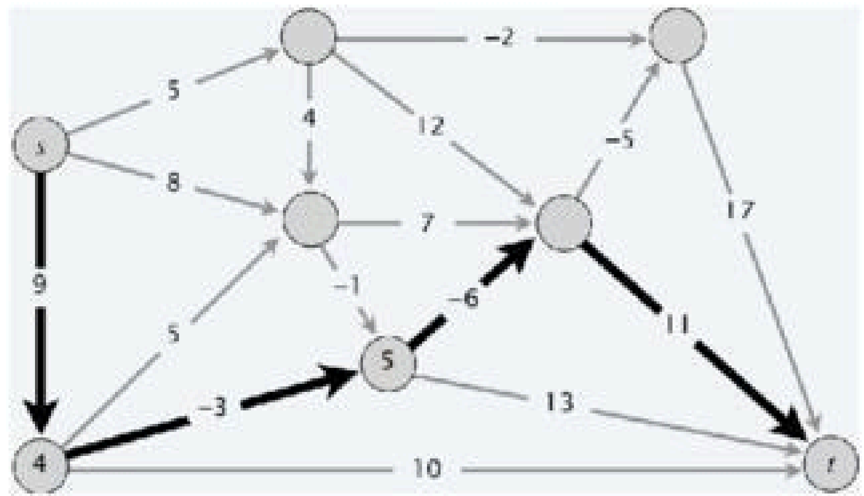
- Gap penalty $\delta = 1$
- Mismatch penalty α
  - $= 0, \text{if } p = q$
  - $= \infty, \text{if } p \neq q$
- Edit distance = # gaps = number of characters deleted from x and y.
- Length of LCS = ($m + n$ – edit distance) / 2.

Analysis. $O(mn)$ time and $O(m + n)$ space.

Bellman–Ford–Moore algorithm

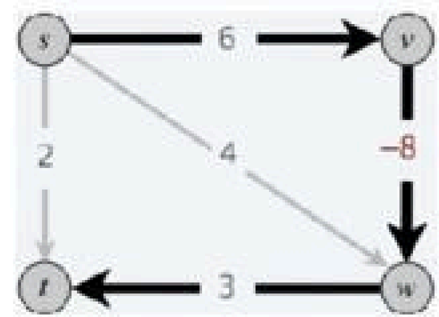
Shortest paths with negative weights

Shortest-path problem. Given a digraph $G = (V, E)$, with arbitrary edge lengths l_{vw}, find shortest path from source node s to destination node t.



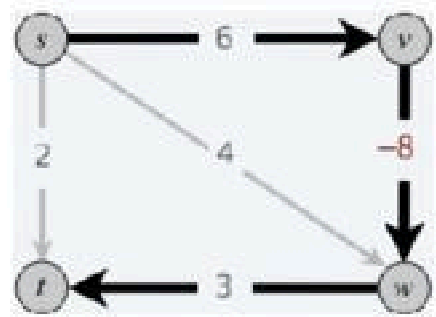
Greedy attempt

Dijkstra. May not produce shortest paths when edge lengths are negative.

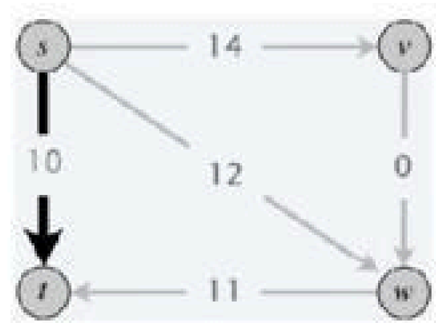


Greedy attempt

Dijkstra. May not produce shortest paths when edge lengths are negative.

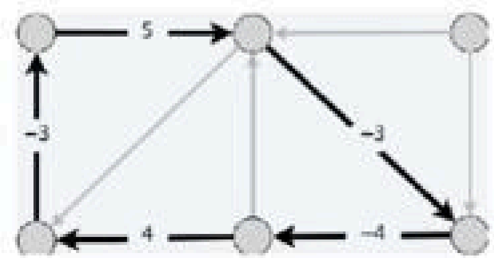


Re-weighting. Adding a constant to every edge length does not necessarily make Dijkstra's algorithm produce shortest paths.



Negative cycles

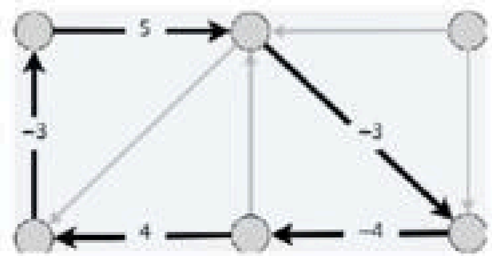
Def. A negative cycle is a directed cycle for which the sum of its edge lengths is negative.



Lemma 1. If some $v \rightsquigarrow t$ path contains a negative cycle, then there does not exist a shortest $v \rightsquigarrow t$ path.
Pf. If there exists such a cycle W, then can build a $v \rightsquigarrow t$ path of arbitrarily negative length by detouring around W as many times as desired.

Negative cycles (cont.)

Def. A negative cycle is a directed cycle for which the sum of its edge lengths is negative.



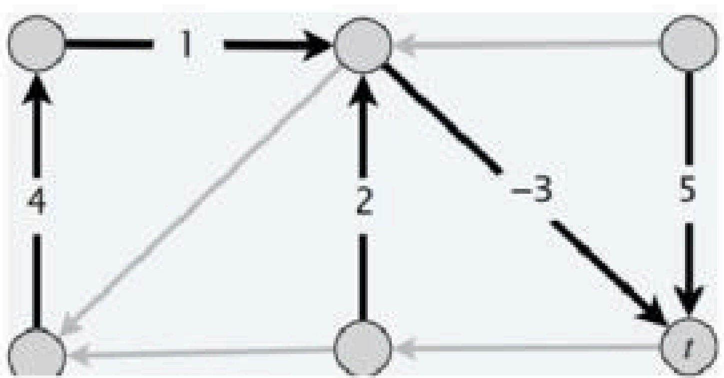
Lemma 2. If G has no negative cycles, then there exists a shortest $v \leadsto t$ path that is simple (and has n – 1 edges).
Pf.

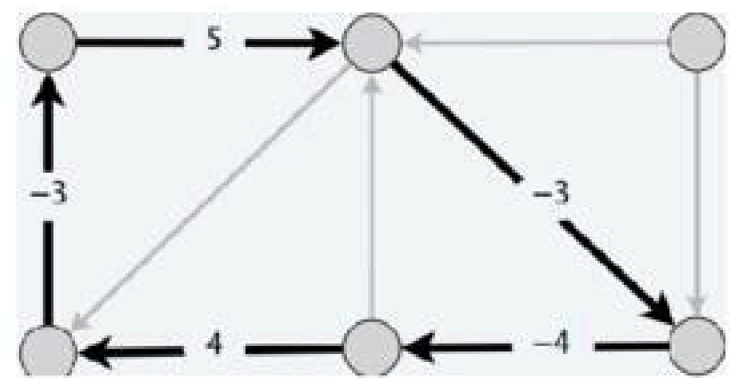
- Among all shortest $v \leadsto t$ paths, consider path P that uses the fewest edges.
- If that path P contains a directed cycle W, can remove the portion of P corresponding to W without increasing its length.

Two problems

Single-destination shortest-paths problem. Given a digraph $G = (V, E)$ with edge lengths l_{vw} (but no negative cycles) and a distinguished node t, find a shortest $v \leadsto t$ path for every node v.

Negative-cycle problem. Given a digraph $G = (V, E)$ with edge lengths l_{vw}, find a negative cycle (if one exists).





Quiz: shortest-paths via DP

Which sub-problems to find shortest $v \rightsquigarrow t$ paths for every node v?

A. $OPT(i, v)$ = length of shortest $v \rightsquigarrow t$ path that uses *exactly* i edges.
B. $OPT(i, v)$ = length of shortest $v \rightsquigarrow t$ path that uses *at most* i edges.
C. Neither A nor B.

Quiz: shortest-paths via DP

Which sub-problems to find shortest $v \leadsto t$ paths for every node v?

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C. Neither A nor B.

A: cannot eliminate shorter paths, since adding a negative edge may greatly reduce length and cancel previous effort, thus reduce to brute-force

DP for shortest-paths

Def. $OPT(i, v)$ = length of shortest $v \rightsquigarrow t$ path that uses $\leq i$ edges.

Goal. $OPT(n-1, v)$ for each v.

- by Lemma 2, simple path has $\leq n-1$ edges.

DP for shortest-paths

Def. $OPT(i, v)$ = length of shortest $v \leadsto t$ path that uses $\leq i$ edges.

Goal. $OPT(n\text{–}1, v)$ for each v.

- by Lemma 2, simple path has $\leq n\text{–}1$ edges.

Case 1. Shortest $v \leadsto t$ path uses $\leq i\text{–}1$ edges.

- $OPT(i, v)$ = $OPT(i\text{–}1, v)$.

Case 2. Shortest $v \leadsto t$ path uses exactly i edges.

- if (v, w) is first edge in shortest such $v \leadsto t$ path, incur a cost of l_{vw}.
- Then, select best $w \leadsto t$ path using $\leq i\text{–}1$ edges.

DP for shortest-paths: Bellman

Def. $OPT(i, v)$ = length of shortest $v \leadsto t$ path that uses $\leq i$ edges.

Goal. $OPT(n\text{–}1, v)$ for each v.

Case 1. Shortest $v \leadsto t$ path uses $\leq i\text{–}1$ edges.

Case 2. Shortest $v \leadsto t$ path uses exactly i edges.

Bellman equation.

$$OPT(i, v) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v \\ \infty & \text{if } i = 0 \text{ and } v \\ \min\{OPT(i-1, v), \min_{(v,w) \in E}\{OPT(i-1, w) + l_{vw}\}\} & \text{if } i > 0 \end{cases}$$

DP for shortest-paths: algorithm

SHORTEST-PATHS(V, E, l, t)

1. FOREACH node $v \in V$: $M[0, v] = \infty$;
2. $M[0, t] = 0$;
3. FOR $i = 1..n{-}1$:
    1. FOREACH node $v \in V$:
        1. $M[i, v] = M[i{-}1, v]$;
        2. FOREACH edge $(v, w) \in E$: $M[i, v] = min\{M[i, v], M[i{-}1, w] + l_{vw}\}$;

$$OPT(i, v) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v \\ \infty & \text{if } i = 0 \text{ and } v \\ \min\{OPT(i-1, v), \min_{(v,w)\in E}\{OPT(i-1, w) + l_{vw}\}\} & \text{if } i > 0 \end{cases}$$

DP for shortest-paths: analysis

Theorem 1. Given a digraph $G = (V, E)$ with no negative cycles, the DP algorithm computes the length of a shortest $v \rightsquigarrow t$ path for every node v in $\Theta(mn)$ time and $\Theta(n^2)$ space.

Pf.

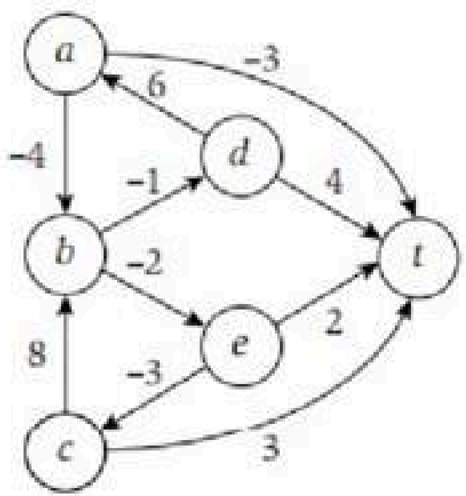
- Table requires $\Theta(n^2)$ space.
- Each iteration i takes $\Theta(m)$ time since we examine each edge once.

≡

DP for shortest-paths: trace-back

Finding the shortest paths.

- Approach 1: Maintain `successor`$[i, v]$ that points to next node on a shortest $v \rightsquigarrow t$ path using $\leq i$ edges.
- Approach 2: Compute optimal lengths $M[i, v]$ and consider only edges with $M[i, v] = M[i{-}1, w] + l_{vw}$. Any directed path in this subgraph is a shortest path.



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
e	∞	2	0	0	0	0

Quiz: DP for shortest-paths

It is easy to modify the DP algorithm for shortest paths to ...

A. Compute lengths of shortest paths in $O(mn)$ time and $O(m+n)$ space.
B. Compute shortest paths in $O(mn)$ time and $O(m+n)$ space.
C. Both A and B.
D. Neither A nor B.

Shortest-paths: practical improvements

Space optimization. Maintain two 1D arrays (instead of 2D array).

- $d[v]$ = length of a shortest $v \rightsquigarrow t$ path that we have found so far.
- `successor`[v] = next node on a $v \rightsquigarrow t$ path.

Performance optimization. If $d[w]$ was not updated in iteration $i-1$, then no reason to consider edges entering w in iteration i.

Bellman–Ford–Moore

BELLMAN–FORD–MOORE(V, E, c, t)

```
FOREACH node v:
   d[v] = INF;
   successor[v] = null;
d[t] = 0;
FOR i = 1 .. n – 1:
   FOREACH node w:
      IF (d[w] was updated in previous pass):
         FOREACH edge (v, w):
            IF (d[v] > d[w] + l vw) d[v] = d[w] + l vw;
```

≡

Quiz: Bellman–Ford–Moore

Which properties must hold after pass i of Bellman–Ford–Moore?

A. $d[v]$ = length of a shortest $v \leadsto t$ path using $\leq i$ edges.
B. $d[v]$ = length of a shortest $v \leadsto t$ path using exactly i edges.
C. Both A and B.
D. Neither A nor B.

Quiz: Bellman–Ford–Moore

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A. $d[v]$ = length of a shortest $v \rightsquigarrow t$ path using $\leq i$ edges.
B. $d[v]$ = length of a shortest $v \rightsquigarrow t$ path using exactly i edges.
C. Both A and B.
D. Neither A nor B.

D. Now i is just a counter of $n - 1$ iterations.

Bellman–Ford–Moore: analysis

Lemma 3. For each node v: $d[v]$ is the length of some $v \rightsquigarrow t$ path.

Lemma 4. For each node v: $d[v]$ is monotone non-increasing.

Bellman–Ford–Moore: analysis

Lemma 3. For each node v: $d[v]$ is the length of some $v \rightsquigarrow t$ path.
Lemma 4. For each node v: $d[v]$ is monotone non-increasing.

Lemma 5. After pass i, $d[v] \leq$ length of a shortest $v \rightsquigarrow t$ path using $\leq i$ edges.
Pf. [by induction on i]

- Base case: $i = 0$; Assume true after pass i.
- Let P be any $v \rightsquigarrow t$ path with $\leq i + 1$ edges.
- Let (v, w) be first edge in P and let P' be subpath from w to t.
- By inductive hypothesis, at the end of pass i, $d[w] \leq l(P')$, because P' is a $w \rightsquigarrow t$ path with $\leq i$ edges.
- After considering edge (v, w) in pass $i + 1$:

$$\begin{aligned} d[v] &\leq l_{vw} + d[w] \\ &\leq l_{vw} + l(P') \\ &= l(P) \end{aligned}$$

Bellman–Ford–Moore: analysis (cont.)

Theorem 2. Assuming no negative cycles, Bellman–Ford–Moore computes the lengths of the shortest $v \rightsquigarrow t$ paths in $O(mn)$ time and $\Theta(n)$ extra space.

Pf. Lemma 2 + Lemma 5.

- shortest path exists and has at most $n - 1$ edges
- after i passes, $d[v] \leq$ length of shortest path that uses $\leq i$ edges

Bellman–Ford–Moore: analysis (cont.)

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Remark. Bellman–Ford–Moore is typically faster in practice.

- Edge (v, w) considered in pass $i + 1$ only if $d[w]$ updated in pass i.
- If shortest path has k edges, then algorithm finds it after $\leq k$ passes.

Quiz: Bellman–Ford–Moore trace-back

Assuming no negative cycles, which properties must hold throughout Bellman–Ford–Moore?

A. Following `successor`$[v]$ pointers gives a directed $v \rightsquigarrow t$ path.

B. If following `successor`$[v]$ pointers gives a directed $v \rightsquigarrow t$ path, *then* the length of that $v \rightsquigarrow t$ path is $d[v]$.

C. Both A and B.

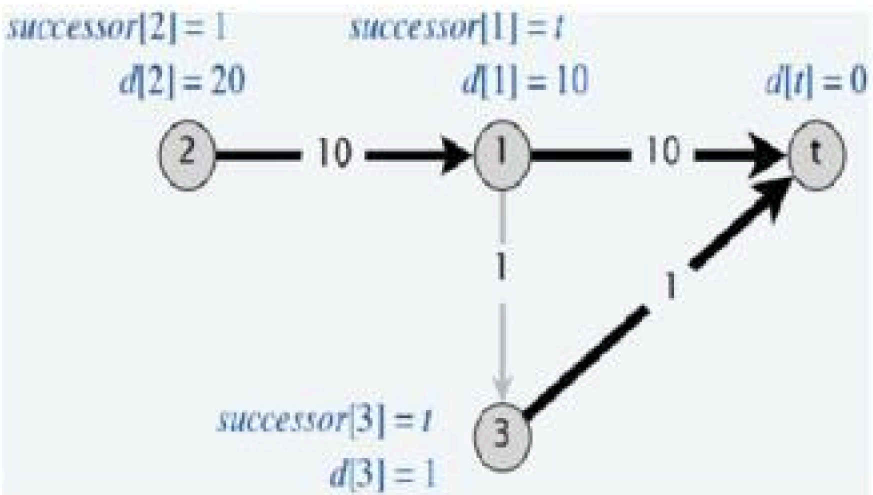
D. Neither A nor B.

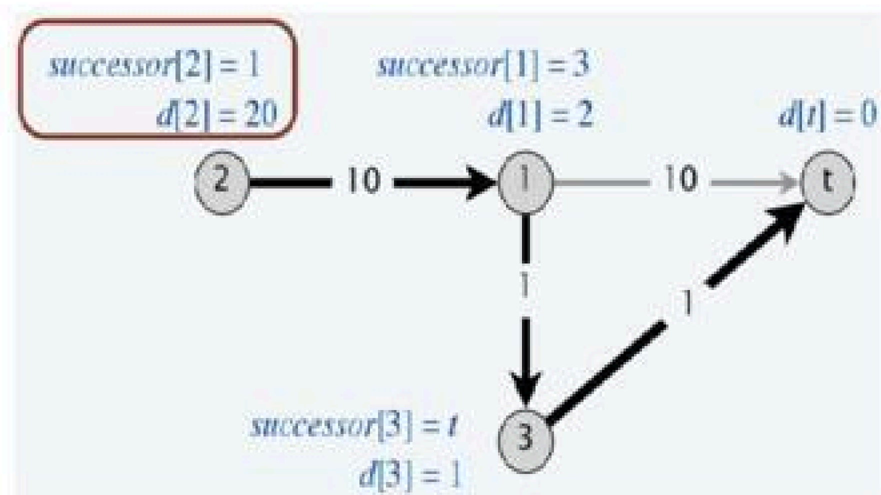
Bellman–Ford–Moore: trace-back

Claim. ~~Throughout Bellman–Ford–Moore, following the `successor`[v] pointers gives a directed path from v to t of length $d[v]$.~~

Counterexample. Claim is false!

- Length of successor $v \rightsquigarrow t$ path may be strictly shorter than $d[v]$.



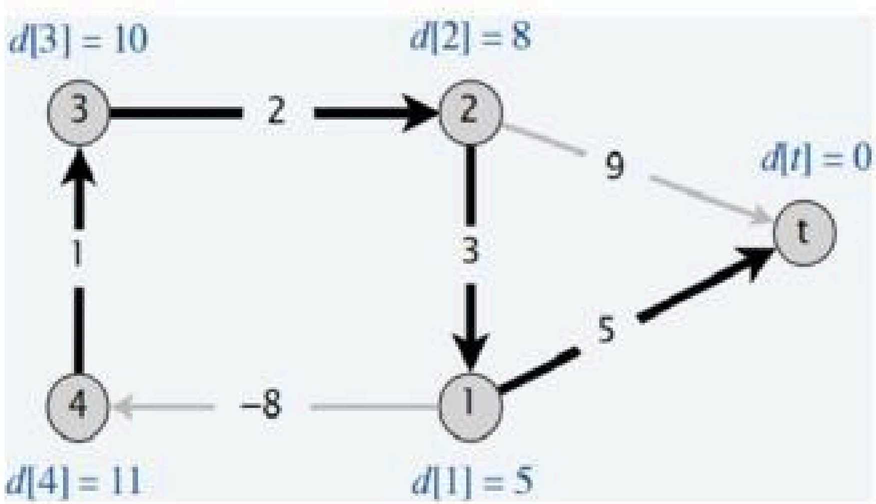


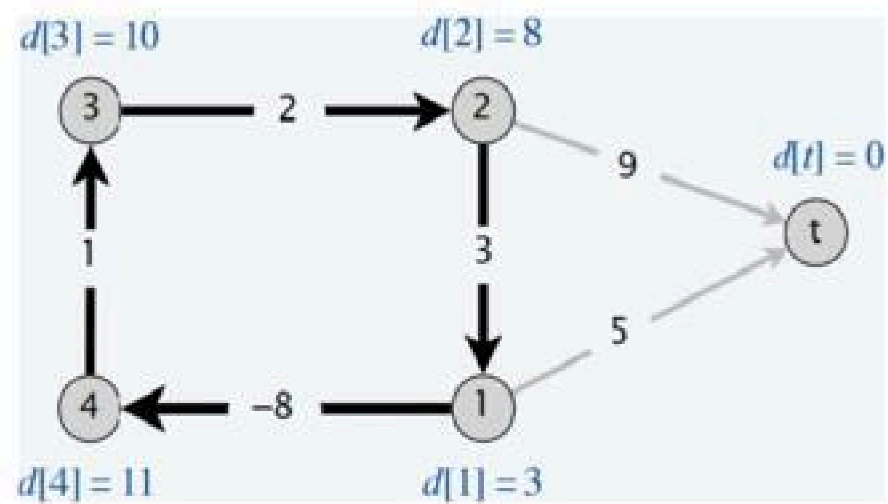
Bellman–Ford–Moore: trace-back

Claim. ~~Throughout Bellman–Ford–Moore, following the~~ successor$[v]$ ~~pointers gives a directed path from v to t of length~~ $d[v]$.

Counterexample. Claim is false!

- Length of successor $v \rightsquigarrow t$ path may be strictly shorter than $d[v]$.
- With negative cycle, successor graph may have directed cycles.





Bellman–Ford–Moore: shortest paths

Lemma 6. Any directed cycle W in the successor graph is a negative cycle.

Pf.

- If `successor`$[v]$ = w, we must have $d[v] \geq d[w] + l_{vw}$.
- Let $v_1 \to v_2 \to \ldots \to v_k \to v_1$ be sequence in a directed cycle W.
- Assume that (v_k, v_1) is the last edge in W added to successor graph.
- Just prior to that:

$$
\begin{aligned}
d[v_1] &\geq l(v_1, v_2) + d[v_2] \\
&\vdots \\
d[v_{k\text{-}1}] &\geq l(v_{k\text{-}1}, v_k) + d[v_k] \\
d[v_k] &> l(v_k, v_1) + d[v_1] \text{ strict less: updating now}
\end{aligned}
$$

- Adding inequalities yields $l(v_1, v_2) + l(v_2, v_3) + \ldots + l(v_{k\text{-}1}, v_k) + l(v_k, v_1) < 0$.

BFM: shortest paths (cont.)

Theorem 3. Assuming no negative cycles, Bellman–Ford–Moore finds shortest $v \rightsquigarrow t$ paths for every node v in $O(mn)$ time and $\Theta(n)$ extra space.

Pf.

- The successor graph cannot have a directed cycle. [Lemma 6]
- Let P: $v = v_1 \to v_2 \to \ldots \to v_k = t$ be following successor pointers.
- Upon termination, if `successor`[v] = w, we must have $d[v] = d[w] + l_{vw}$.

$$
\begin{aligned}
d[v_1] &= l(v_1, v_2) + d[v_2] \\
d[v_2] &= l(v_2, v_3) + d[v_3] \\
&\vdots \\
d[v_{k\text{-}1}] &= l(v_{k\text{-}1}, v_k) + d[v_k]
\end{aligned}
$$

- Adding equations yields $d[v] = d[t] + l(v_1, v_2) + l(v_2, v_3) + \ldots + l(v_{k\text{-}1}, v_k)$.

Distance-vector protocols

Communication network

Communication network.

- Node ≈ router.
- Edge ≈ direct communication link.
- Length of edge ≈ latency of link.

Dijkstra's algorithm. Requires global information of network.

Bellman–Ford–Moore. Uses only local knowledge of neighboring nodes.

Synchronization. We don't expect routers to run in lockstep.

- The order in which each edges are processed in Bellman–Ford–Moore is not important.
- Moreover, algorithm converges even if updates are asynchronous.

Distance-vector routing protocols

Distance-vector routing protocols. ["routing by rumor"]

- Each router maintains a vector of shortest-path lengths to every other node (**distances**) and the *first hop* on each path (**directions**).
- **Algorithm**: each router performs n separate computations, one for each potential destination node.

Ex. Routing Information Protocol (RIP), Xerox XNS RIP, Novell's IPX RIP, Cisco's IGRP, DEC's DNA Phase IV, AppleTalk's RTMP.

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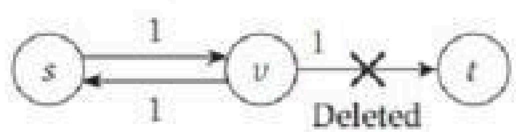
Caveat. Edge lengths may *change* during algorithm (or fail completely).



Path-vector routing protocols

Link-state routing protocols.

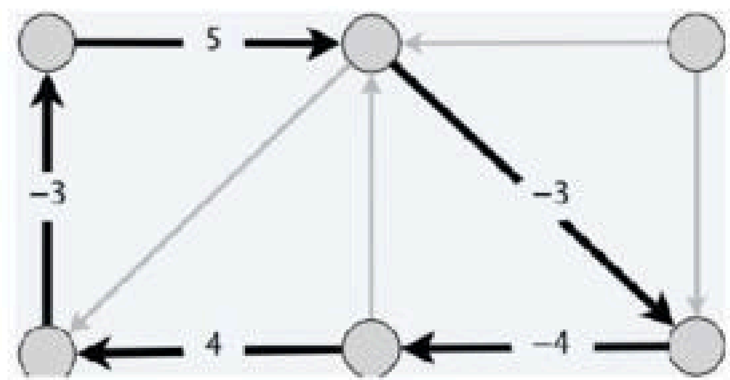
- Each router stores the *whole path* (or network topology).
- Based on Dijkstra's algorithm.
- Avoids "counting-to-infinity" problem and related difficulties.
- Requires significantly more storage.

Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).

Negative cycles

Detecting negative cycles

Negative cycle detection problem. Given a digraph $G = (V, E)$, with edge lengths l_{vw}, find a negative cycle (if one exists).



Detecting negative cycles: application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!

Detecting negative cycles - I

Lemma 7. If $OPT(n, v) = OPT(n\text{–}1, v)$ for every node v, then no negative cycles.
Pf. The $OPT(n, v)$ values have converged $\Rightarrow$ shortest $v \rightsquigarrow t$ path exists.

Lemma 8. If $OPT(n, v) < OPT(n\text{–}1, v)$ for some node v, then (any) shortest $v \rightsquigarrow t$ path of length $\leq n$ contains a cycle W. Moreover W is a negative cycle.
Pf. [by contradiction]

- Since $OPT(n, v) < OPT(n\text{–}1, v)$, we know that shortest $v \rightsquigarrow t$ path P has exactly n edges.
- By *pigeonhole principle*, the path P must contain a repeated node x.
- Let W be any cycle in P.
- Deleting W yields a $v \rightsquigarrow t$ path with $< n$ edges $\Rightarrow$ W is a negative cycle.

Detecting negative cycles - II

Theorem 4. Can find a negative cycle in $\Theta(mn)$ time and $\Theta(n2)$ space.
Pf.

Construct **Augmented graph** G': Add new sink node t and connect all nodes to t with 0-length edge.

- G has a negative cycle iff G' has a negative cycle.

Case 1. [$OPT(n, v) = OPT(n–1, v)$ for every node v]

- By Lemma 7, no negative cycles.

Case 2. [$OPT(n, v) < OPT(n–1, v)$ for some node v]

- Using proof of Lemma 8, can extract negative cycle from $v \rightsquigarrow t$ path. (cycle cannot contain t since no edge leaves t)

≡

Detecting negative cycles - III

Theorem 5. Can find a negative cycle in $O(mn)$ time and $O(n)$ extra space.

Pf.

- Run `Bellman-Ford-Moore` on G' for $n' = n + 1$ passes (instead of n'–1).
- If no $d[v]$ values updated in pass n', then no negative cycles.
- Otherwise, suppose $d[s]$ updated in pass n'.
- Define `pass`(v) = last pass in which $d[v]$ was updated.
- Observe `pass`(s) = n' and `pass`(`successor`[v]) $\geq$ `pass`(v) –1 for each v.
- Following successor pointers, we must eventually repeat a node.
- Lemma 6 $\Rightarrow$ the corresponding cycle is a negative cycle.