Algorithm II

2. Algorithm Analysis

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Why analyzing algorithms?

Precise **assessment** leads to better **understanding**.

- *correctness*
  - theoretical proof
  - practical implementation
- *efficiency*: iterative development
  - computable?
  - what design to choose?
  - any room for improvement? or terminate?

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We focus on the *efficiency* of algorithms now.

Content

- Computational Tractability
- Asymptotic Order of Growth
- Implement Gale–Shapley
- Common Running Times
- Recap: Priority Queue

Computational Tractability

What is “Computational Tractability”

Loosely speaking: delimitate whether a problem can be solved *in practice*.

- usually, relative to current computing power.
  - imagine a cart driven by a motor
- also, contextual tolerance is often a key consideration.
  - e.g., patience of your customer

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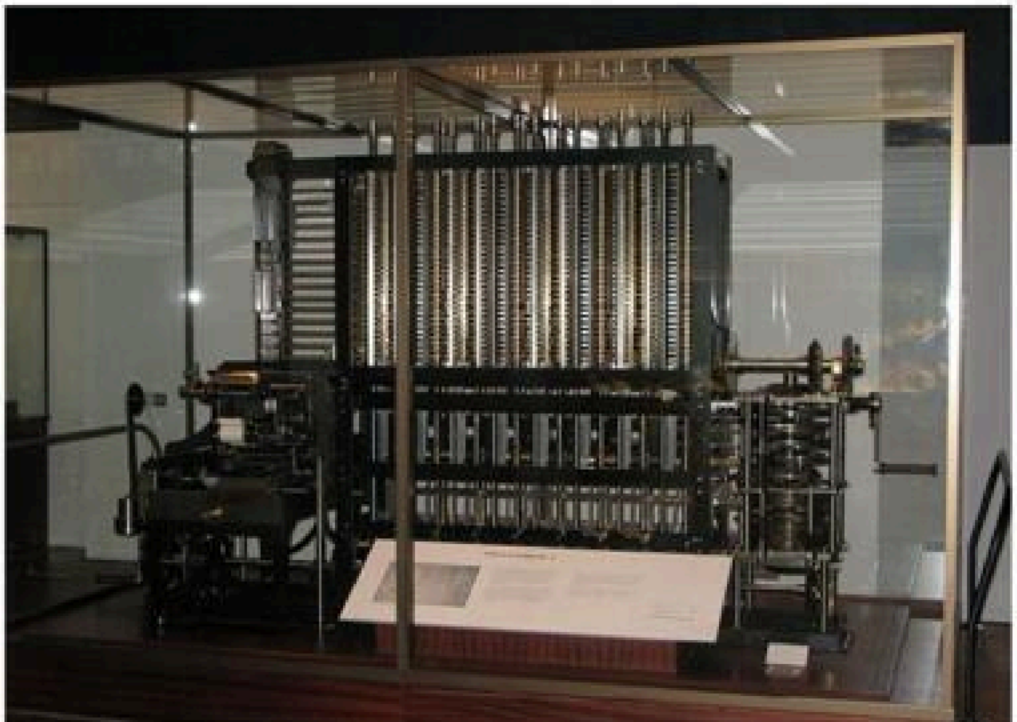
So **efficiency** is about: resource requirements vs. computational power.

Analytic Engine

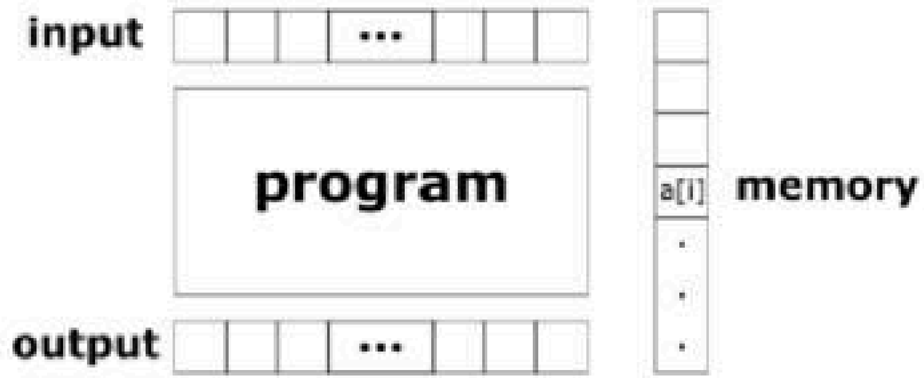
"By what course of calculation can these results be arrived at by the machine in the shortest time?" — Charles Babbage (1864)



Modern computing model

Consider a 64-bit system:

- Each **memory cell** stores a 64-bit integer.
- **Primitive operations**: arithmetic/logic operations, read/write memory, array indexing, following a pointer, conditional branch, etc.



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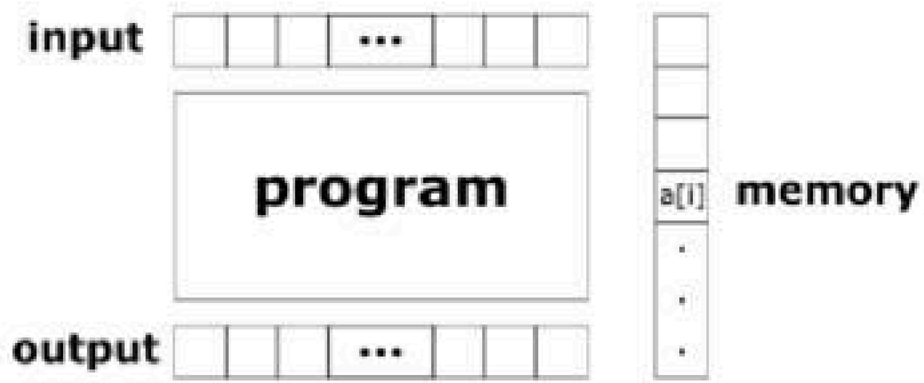

- **Time**: Number of primitive operations, given CPU speed.
- **Space**: Number of memory cells utilized.

How to define efficiency?

Intuition. When implemented, *runs fast and uses few memory* on real inputs.

- what platform? PC, cellphone
- what is a "real" inputs? `struct`, `int`

We need a measure of algorithm *itself*, rather than external indicators.

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- equal: count number of operations/cells required *per unit input*.
  - counter-example: print N number pairs vs. N numbers.

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Better measure: How is the algorithm *scale* with problem size.

Scalability

How resource requirements *grow with increasing input size.*

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So we study and compare *growth of functions*.

- **sampling**: measure efficiency at a series of fixed input numbers.
- compare: “standard” behavior among all possible inputs
  - sorting does nothing (thus fast), when input already sorted

Worst-Case Analysis

Worst-Case Running Times: longest possible running time.

- well-accepted standard, but *not perfect*
  - pathological inputs can lead to bad performance
  - hard to find effective alternative

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Average-case analysis: averaged over "random" instances.

- more about how random inputs were generated (than algorithm itself)
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Now consider and compare $T(N)$ on worst-cases

- need a *baseline* implementation to mark the worst possibility.

Brute-Force Search

Brute-Force Search: the most natural last-resort solution.

- enumerate all possibilities
  - no use in practice, but usually gives *exact analytical bounds*.
  - Stable matching: test all $n!$ perfect matchings for stability.

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What is "qualitatively better"? Better scalability

- brute-force search usually grow *exponentially fast*
- intuitively, growth rate should be much slower

Polynomial running time

Desirable scaling property. When input size *doubles*, algorithm slow down by at most some multiplicative constant factor C.

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An algorithm is poly-time if the above scaling property holds.

There exist constants $c > 0$ and $d > 0$ such that, for every input of size N, the running time of the algorithm is *bounded above* by cN^d primitive computational steps.

- here $C = 2^d$
- lower-degree polynomials grow slower

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- becomes negatable: define **inefficiency**

Exceptions: galactic constants and/or huge exponents

- which is better: $20n^{120}$ or $n^{1+0.02\ln n}$?

≡

Common polynomials

Assume: one million (10^6) high-level instructions per second.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Notice the the huge difference between polynomial and exponential.

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Now we found the way to compare growth of functions.

- compare different *categories* of growth rate

Asymptotic Order of Growth

Asymptotic analysis

Mathematically, asymptotic is used for describing *limiting* behavior.

- rigorous description of scalability: growth rate
- only a coarser level of granularity is necessary
  - Ex. $1.62n^2 + 3.5n + 8$ steps

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Caution. In CS, deal with discrete quantities.

- no such thing as "infinitesimal" in calculus.

Asymptotic Upper Bounds (Big O)

$T(n)$ is $O(f(n))$ (read as "$T(n)$ is order $f(n)$")

- for sufficiently large n, function $T(n)$ is *bounded above* by a constant multiple of $f(n)$.
- $\exists c > 0, n_0 \geq 0 : \forall n \geq n_0, T(n) \leq cf(n)$.
  - c cannot depend on n.

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Ex. $T(n) = pn^2 + qn + r$:

- $T(n) = pn^2 + qn + r \leq pn^2 + qn^2 + rn^2 = (p + q + r)n^2$
  - $T(n) \leq cn^2 \in O(n^2)$, where $c = p + q + r$.

Big O notational abuses

One-way "equality". $O(g(n))$ is a set of functions.

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- $T(n) = pn^2 + q^n + r = O(n^3)$, since $n^2 \leq n^3$.
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Domain and Range. T and f are real-valued functions.

- domain is typically natural numbers: $\mathbb{N} \to \mathbb{R}$.
- Sometimes extend to the reals: $\mathbb{R}_{\geq 0} \to \mathbb{R}$.
- Or restrict to a subset.

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Products. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1\ f_2$ is $O(g_1 g_2)$.
Pf.

- $\exists c_1 > 0 and n_1 \geq 0$ such that $0 \leq f_1(n) \leq c_1 g_1(n)$ for all $n \geq n_1$.
- $\exists c_2 > 0 and n_2 \geq 0$ such that $0 \leq f_2(n) \leq c_2 g_2(n)$ for all $n \geq n_2$.
- Then, $0 \leq f_1(n) f_2(n) \leq c_1 c_2 g_1(n) g_2(n)$ for all $n \geq \max\{n_1, n_2\}$.

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Transitivity. If f is $O(g)$ and g is $O(h)$, then f is $O(h)$.

Ex. $f(n) = 5n^3 + 3n^2 + n + 1234$ is $O(n^3)$.

Asymptotic Lower Bounds (Big Ω)

$T(n)$ is $\Omega(f(n))$ ("$T(n) = \Omega(f(n))$")

- for sufficiently large n, function $T(n)$ is *at least* a constant multiple of $f(n)$.
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Ex. $T(n) = 32n^2 + 17n + 1$

- $T(n)$ is both $\Omega(n^2)$ and $\Omega(n)$.
- $T(n)$ is not $\Omega(n^3)$.

Asymptotically Tight Bounds (Big Θ)

$T(n)$ is $\Theta(f(n))$ ("$T(n) = \Theta(f(n))$")

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- $\exists c_1 > 0, c_2 > 0, n_0 \geq 0 : \forall n \geq n_0, 0 \leq c_1 f(n) \leq T(n) \leq c_2 f(n)$.
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Compute: closing *gap* between upper bound and lower bound

- design: *a* worst-case algorithm as upper bound
  - prove: no better possibilities
- justify asymptotically *tight* bound on worst-case running time

Asymptotic bounds and limits

Proposition. If for some constant $0 < c < \infty$ $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$ then $f(n)$ is $\Theta(g(n))$

.

Pf.

- By definition of the limit, $\forall \epsilon > 0, \exists n_0 : c - \epsilon \leq \frac{f(n)}{g(n)} \leq c + \epsilon, \forall n \geq n_0$.
- Choose $\epsilon = \frac{1}{2}c > 0$.
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Asymptotic bounds for common functions

Polynomials. Let $f(n) = a_0 + a_1 n + \ldots + a_d n^d$ with $a_d > 0$. Then $f(n) = \Theta(n^d)$.

Pf. $\lim_{n \to \infty} \frac{a_0 + a_1 n + \ldots + a_d n^d}{n^d} = a_d > 0$

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Logarithms. $\log_a n = \Theta(\log_b n)$ for every $a > 1$ and every $b > 1$.

Pf. $\frac{\log_a n}{\log_b n} = \frac{1}{\log_b a}$.

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Logarithms and polynomials. $\log_a n = O(n^d)$ for every $a > 1$ and every $d > 0$.

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Pf. $\lim_{n \to \infty} \frac{\log_a n}{n^d} = 0$.

Exponentials and polynomials. $n^d = O(r^n)$ for every $r > 1$ and every $d > 0$.

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Asymptotic bounds for common functions

Polynomials. Let $f(n) = a_0 + a_1 n + \ldots + a_d n^d$ with $a_d > 0$. Then $f(n) = \Theta(n^d)$.

Pf. $\lim_{n \to \infty} \frac{a_0 + a_1 n + \ldots + a_d n^d}{n^d} = a_d > 0$

Logarithms. $\log_a n = \Theta(\log_b n)$ for every $a > 1$ and every $b > 1$.

Pf. $\frac{\log_a n}{\log_b n} = \frac{1}{\log_b a}$.

Logarithms and polynomials. $\log_a n = O(n^d)$ for every $a > 1$ and every $d > 0$.

Pf. $\lim_{n \to \infty} \frac{\log_a n}{n^d} = 0$.

Exponentials and polynomials. $n^d = O(r^n)$ for every $r > 1$ and every $d > 0$.

Pf. $\lim_{n \to \infty} \frac{n^d}{r^n} = 0$.

Factorials. $n! = 2^{\Theta(n \log n)}$.

Pf. Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Big O notation with multiple variables

Upper bounds. $f(m, n) = O(g(m, n))$ if there exist constants $c > 0, m_0 \geq 0$, and $n_0 \geq 0$ such that $0 \leq f(m, n) \leq cg(m, n)$ for all $n \geq n_0, m \geq m_0$.

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Ex. $f(m,n) = 32mn^2 + 17mn + 32n^3$.

- $f(m,n)$ is both $O(mn^2 + n^3)$ and $O(mn^3)$.
- $f(m,n)$ is neither $O(n^3)$ nor $O(mn^2)$.
  - $f(m,n)$ is $O(n^3)$ if a precondition to the problem implies $m \leq n$.

Implement Gale–Shapley

Goal: $O(n^2)$ implementation

Compute: closing *gap* between upper bound and lower bound

- design: *a* worst-case algorithm as upper bound
  - prove: no better possibilities

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Recall: Algorithm terminates in at most n^2 iterations

- worst-case bound: $O(n^2)$
  - **Goal**: find a $O(n^2)$ implementation
    - each iteration takes constant time, ie., $O(1)$
- tightest? discuss later

Recap: Gale–Shapley

INPUT: M, W, R_m, R_w

1. $P = \varnothing$; mark $m \in M$ and $w \in W$ `free`;
2. WHILE some $m \in M$ is `free`
   1. w: highest on R_m that m has not yet proposed;
   2. IF w is `free`
      1. Add (m, w) to P;
   3. ELSE IF w prefers m to current partner m'
      1. Replace (m', w) with (m, w), set m' `free`;
   4. ELSE (Nothing happens.);
3. RETURN P;

Constant time operations

Goal: the following operations take constant time:

1. identify a `free` m.
2. given m, identify highest-ranked w that m not yet proposed.
3. given w, decide if is `matched`,
   - if so, identify current partner m'.
4. identify which ranks higher for w: m or m'.

Representation 1: next `free`

List NF containing indices of M (queue or stack also works)

- initialize to n indices
  - initialization: $O(n)$ (is also $O(n^2)$)
- take next free element: $O(1)$
- if replaced, push back: $O(1)$

Representation 2: proposal

Reuse the preference list R_m, only check the head.

- head always has highest-ranked w: $O(1)$
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If R_m is `constant`, maintain a pointer to next proposal.

- moving pointer to the next: $O(1)$

Representation 3: matching

Index $M, W: \{1, 2, \ldots, n\}$.

Matching. Arrays P_m, P_w.

- if m matched to w: $P_m(m) = w, P_w(w) = m$.
  - add/remove matching pair: $O(1)$
- initialize P_m, P_w to 0: `unmatched`.
  - identify whether `matched`, and to whom: $O(1)$
  - initialization: $O(n)$ (= $O(n^2)$)

Representation 4: compare ranks

So far, operation 1-3 can be implemented in $O(1)$ time.

- now: identify which ranks higher for w: m or m'.

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Naive implementation: walk R_w

- $O(n)$ time to find m and m' on the list
  - breaks our $O(1)$ time goal

Alternative: trade space for time.

Representation 4: compare ranks (cont.)

For each $w \in W$, maintain an array I_w contains the inverse of R_w.

pref[]	1st	2nd	3rd	4th	5th	6th	7th	8th
	8	3	7	1	4	5	6	2

rank[]	1	2	3	4	5	6	7	8
	4th	8th	2nd	5th	6th	7th	3rd	1st

- for i = 1 to n: $I_w[R_w[i]] = i$
  - only need to compare $I_w[m]$ and $I_w[m']$: $O(1)$
- $\Theta(n^2)$ time initialization: iterate for each w.

Gale–Shapley implementation: summary

Theorem. Can implement Gale–Shapley to run in $O(n^2)$ time.

Pf.

- $\Theta(n^2)$ preprocessing time to create n inverse ranking arrays.
- There are $O(n^2)$ proposals; processing each proposal takes $O(1)$ time.

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Conclusion. $GS = \Theta(n^2)$

Common Running Times

Constant time

Constant time. Running time is $O(1)$.

- bounded by constant: not depend on input size n

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Examples.

- Conditional branch.
- Arithmetic/logic operation.
- Declare/initialize a variable.
- Follow a link in a linked list.
- Access element i in an array.
- Compare/exchange two elements in an array.

Linear time

Linear time. Running time is $O(n)$.

- process input in a single pass,
  - spending constant time on each encountered item

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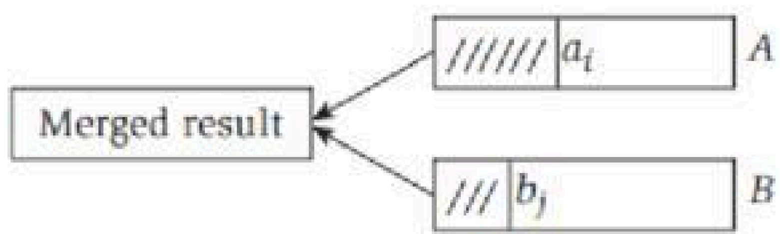
Computing the Maximum.

Merge two sorted lists. Combine two *sorted* linked lists $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$ into a sorted whole.

- at most $2n$ iterations



Quiz: Target-Sum

Target-Sum. Given a *sorted* array of n distinct integers and an integer T, find two that sum to exactly T?

Hint: move two indices from opposite side towards each other.

Logarithmic time (Sublinear)

Logarithmic time. Running time is $O(\log n)$.

- splits input into two equal-sized pieces,
  - solves each piece recursively,
  - then combines two solutions in *constant time*.
- divide-and-conquer

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Search in a sorted array. Given a sorted array A of n distinct integers and an integer x, find index of x in array.
Binary search.

- Invariant: If x is in the array, then x is in $A[lo..hi]$.
- After k iterations of `WHILE` loop, $(hi - lo + 1) \leq n/2^k \Rightarrow k \leq 1 + \log_2 n$.

Demo: Binary search

Linearithmic time

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Sorting. Given an array of n elements, rearrange them in ascending order.

Merge sort

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Quadratic time

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- nested loops: search over all pairs of input items
  - spend constant time per pair.
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Closest pair of points. Given a list of n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, find the pair that is closest to each other.
$O(n^2)$ **algorithm**. Enumerate all pairs of points (with $i < j$).

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion.

Cubic time

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- nested loops: search over all subsets of size 3.
- almost the borderline of practical

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3-SUM. Given an array of n distinct integers, find three that sum to 0.
$O(n^3)$ **algorithm**. Enumerate all triples (with $i < j < k$).

Remark. $\Omega(n^3)$ seems inevitable, but $O(n^2)$ is not hard.

Polynomial time

Polynomial time. Running time is $O(n^k)$ for some constant $k > 0$.

- nested loops: search over all subsets of size k.
- computationally too hard to be practical

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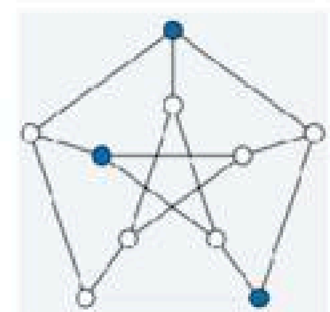
Independent set of size k. Given a graph, find k nodes such that no two are joined by an edge.
$O(n^k)$ **algorithm**. Enumerate all subsets of k nodes.



- Check whether S is an independent set of size k takes $O(k^2)$ time.
- Number of k-element subsets = $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1} \leq \frac{n^k}{k!}$
- in total: $O(k^2 n^k / k!) = O(n^k)$

Exponential time

Exponential time. Running time is $O(2^{n^k})$ for some constant $k > 0$.

- combinatorial: enumerate all subsets

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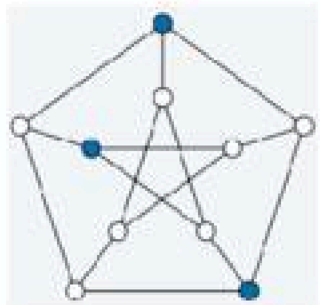
Independent set. Given a graph, find independent set of max cardinality.
$O(n^2 2^n)$ **algorithm**. Enumerate all subsets of n elements.



- total number of subsets: 2^n

Quiz: Exponential time

Which is an equivalent definition of exponential time?

- $O(2^n)$.
- $O(2^{cn})$ for some constant $c > 0$.
- Both.
- Neither.

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Neither: take the limit of division.

Recap: Priority Queue

Priority

Primary goal. seek algorithms that improve qualitatively on brute-force search.

- use polynomial-time solvability as concrete formulation
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Priority Queue. Each element has a priority value.

- properties
  - always take out the highest-priority element
  - $O(\log n)$ time per operation.
- should be familiar after taking *Data Structure* & *Operating System*.

Heap

Maintain elements in sorted order of keys.

- alternatives: sorted array, sorted doubly linked list

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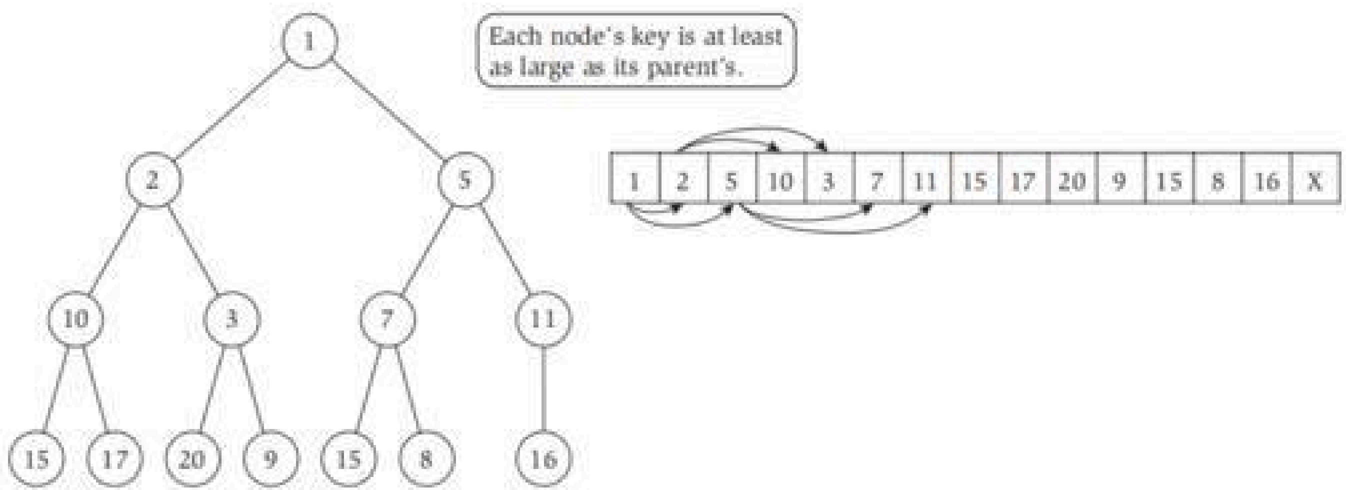
Maintain elements in sorted order of keys.

- alternatives: sorted array, sorted doubly linked list

Conceptually, think heap as *balanced binary tree*

Heap order: For every element v, at a node i, the element w at i's parent satisfies $key(w) \leq key(v)$.



Heap Operations

Heapify-up: fixing the heap by pushing the damaged part upward.

- insert a new element in a heap of n elements in $O(\log n)$ time.

Heapify-down: proceeds down the tree recursively.

- delete a new element in a heap of n elements in $O(\log n)$ time.

Implementing Priority Queues

- StartHeap(N): $O(n)$
- Insert(H, v): $O(\log n)$
- FindMin(H): $O(1)$
- Delete(H, i): $O(\log n)$
- ExtractMin(H): $O(\log n)$